FINITE OLIVER GROUPS WITH RANK INTEGER 0 OR 1 AND SMITH EQUIVALENCE OF GROUP MODULES

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Abstract. Our main algebraic theorem presents a classification of finite Oliver groups G with the rank integer $r_G = 0$ or 1. The famous Smith isomorphism question reads as follows. If a finite group G acts smoothly on a sphere with exactly two fixed points, is it true that the tangent G-modules at the two points are always isomorphic? We show that this question has the negative answer and $r_G \geq 2$ for any finite Oliver group G of odd order, and for any finite Oliver group G with a cyclic quotient of order pq for two distict odd primes p and q. We also show that with just one unknown case, this question has the negative answer for any finite nonsolvable gap group G with $r_G \geq 2$. Moreover, we deduce that for a finite simple group G, the answer to the Smith isomorphism question is affirmative if and only if $r_G = 0$ or 1.

Introduction

Let G be a finite group. For convenience, we say that a series $P \trianglelefteq H \trianglelefteq G$ of normal subgroups H of G and P of H is an *isthmus series* if P and G/H are of prime power order and H/P is cyclic. It turns out that the following three statements are equivalent.

(1) G has a smooth action on a disk without fixed points.

(2) G has no isthmus series of subgroups.

(3) G has a smooth action on a sphere with exactly one fixed point.

In fact, by the work of Oliver [O1], (1) and (2) are equivalent. By the Slice Theorem, (3) implies (1), and according to Laitinen and Morimoto [LM], (2) implies (3).

Following [LM], a finite group G is called an *Oliver group* if G has no isthmus series of subgroups. Recall that each finite nonsolvable group G is an Oliver group, and a finite abelian (more generally, nilpotent) group G is an Oliver group if and only if G has three or more noncyclic Sylow subgroups (cf. [O1], [O2], and [LMP]).

Key words and phrases. Finite Oliver group, rank integer of a finite group G, smooth G-action on disk or sphere, tangent G-module, Oliver or Smith equivalence of G-modules.

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Following [LP], two elements g and h of a finite group G are called *real conjugate* in G, written $g \overset{\pm 1}{\sim} h$, if h is conjugate to g or g^{-1}. Clearly, the relation $\overset{\pm 1}{\sim}$ in G is an equivalence relation. The resulting equivalence class $(g)^{\pm 1}$ of g is called the *real conjugacy class* of g. It follows that $(g)^{\pm 1} = (g) \cup (g^{-1})$, the union of the conjugacy classes (g) and (g^{-1}).

Let G be a finite group. By a real G*-module* we mean a finite dimensional real vector space V with a linear action of G. Let $RO(G)$ be the real representation ring of G and let $IO(G)$ be the intersection of the kernels of the restriction maps $RO(G) \to RO(P)$ taken for all subgroups P of G of prime power order. We put $PO(G) = IO(G) \cap \operatorname{Ker}(RO(G) \to \mathbb{Z})$, where the map $RO(G) \to \mathbb{Z}$ is defined by mapping the difference $U - V \in RO(G)$ into the difference $\dim U^G - \dim V^G \in \mathbb{Z}$ for any two real G-modules U and V.

Following [LP], we denote by r_G the number of the real conjugacy classes represented by elements of G which are not of prime power order. The ranks of the free abelian groups $IO(G)$ and $PO(G)$ are computed in [LP, Lemma 2.1] to the effect that $\operatorname{rk} IO(G) = r_G$ and $\operatorname{rk} PO(G) = r_G - 1$ for $r_G \geq 1$, and $IO(G) = PO(G) = 0$ for $r_G = 0$. Therefore $PO(G) = 0$ if and only if $r_G = 0$ or 1. We refer to r_G as to the *rank integer* of G.

For a smooth action of a finite group G on a smooth manifold M, take the derivatives of the transformations $g : M \to M$ determined by the elements $g \in G$. For any point $x \in M$ left fixed by the action of G, this allows us to consider the tangent space $T_x(M)$ as a real G-module which we refer to as to the *tangent* G*-module at* x.

In 1960, in the case where G is a finite group, Paul A. Smith [Sm, page 406] asked the following question which is still not answered completely.

Smith isomorphism question. *Is it true that for any smooth action of* G *on a sphere with exactly two fixed points, the tangent* G*-modules at the two points are isomorphic?*

Following [Pe1]–[Pe4], given a finite group G, two real G-modules U and V are called *Smith equivalent* if there exists a smooth action of G on a sphere S^n with exactly two fixed points x and y, such that as real G-modules, $T_x(S^n) \cong U$ and $T_y(S^n) \cong V$.

We denote by $Sm(G)$ the subset of $RO(G)$ which is formed by taking the differences $U - V$ represented by two Smith equivalent real G-modules U and V. It is an open question whether $Sm(G)$ is a subgroup of $RO(G)$. However, for a real G-module V,

$$0 = V - V = (V - V^G) - (V - V^G) \in Sm(G).$$

So, for any finite group G, $Sm(G)$ contains the trivial subgroup 0 of $RO(G)$ and the Smith isomorphism question can be restated as follows. *Is it true that* $Sm(G) = 0$ *?*

By [AB], [Mi], [Sa], $Sm(\mathbb{Z}_{p^a}) = 0$ for any odd prime p and any integer $a \geq 1$, and by character theory, $Sm(S_3) = 0$ and $Sm(\mathbb{Z}_n) = 0$ for $n = 2$, 4, or 6. On the other hand, by [CS1]–[CS3], $Sm(\mathbb{Z}_n) \neq 0$ for $n = 4q$ with $q \geq 2$. Hence, $G = \mathbb{Z}_8$ is the smallest group such that $Sm(G) \neq 0$. We refer the reader to [AB], [CS1]–[CS3], [Ch], [DP], [DPS], [DS], [DW], [I], [LP], [MPe], [Mi], [Pa1], [Pa3], [Pe1]–[Pe4], [PR], [Sa], [Sch], [Suh] for more information related with Smith equivalence of G-modules. In this paper, we describe large classes of groups G such that the following two conjectures about $Sm(G)$ are true.

Conjecture 1. $Sm(G) = 0$ for any finite Oliver group G with $r_G \leq 1$.

Conjecture 2. $Sm(G) \neq 0$ for any finite Oliver group G with $r_G \geq 2$.

For a finite group G, we set $Pm(G) = PO(G) \cap Sm(G)$. Recall that $PO(G) = 0$ if and only if $r_G \leq 1$. In effect, $Pm(G) = 0$ for $r_G \leq 1$. For specific finite Oliver groups G with $r_G \geq 2$, we prove that $Pm(G) \neq 0$ and thus $Sm(G) \neq 0$, confirming Conjecture 2.

Here, we wish to mention that for a finite Oliver group G with $r_G \geq 2$, the claim that $Pm(G) \neq 0$ is true once so is the Laitinen Conjecture posed in [LP, Appendix].

Our main algebraic theorem (Classification Theorem) allows us to obtain information (Theorems A1–A3) about a subgroup $LO(G)$ of $PO(G)$. Our main topological theorem (Realization Theorem) and Theorems A1–A3 allow us to get Theorems B1–B3 and C1–C3 which confirm Conjectures 1 and 2 for large classes of finite Oliver groups G.

In particular, contrary to the speculation in [Sch, Comment (2), p. 547] that $Sm(G) \neq 0$ for any finite Oliver group G, we show that there exist precisely fourteen finite nonabelian simple groups G such that $Sm(G) = 0$ (Theorem C1). We recall that any finite nonabelian simple group G is a nonsolvable (and thus Oliver) group.

Moreover, we give counterexamples to the conjecture in [DS, p. 44] that $Sm(H) = 0$ for any subgroup H of a finite group G with $Sm(G) = 0$. In fact, we show that there exist precisely four finite nonabelian simple groups G such that $Sm(G) = 0$ and G has a cyclic subgroup H of order 8 (Theorem C1), and $Sm(H) \neq 0$ by [CS1]–[CS3].

Now, we wish to state the main results of this paper. First, we present a classification of finite Oliver groups G with rank integer $r_G \leq 1$. Below, we deal with the following groups: cyclic groups C_q of order q and dihedral groups D_q of order $2q$, finite elementary abelian p-groups $C_p^k = C_p \times \cdots \times C_p$ (k-times), alternating groups A_n, symmetric groups S_n, matrix groups $GL(n,q)$, $SL(n,q)$, $PGL(n,q)$, $PSL(n,q)$, $PSU(n,q)$, the Mathieu groups M_{10} (the 1-point stabilizer in M_{11}), M_{11}, M_{22}, and the Suzuki groups $Sz(8)$, $Sz(32)$.

Also, $P\Sigma L(n,q)$ and $\Sigma L(n,q)$ denote the splitting extensions of $PSL(n,q)$ and $SL(n,q)$, respectively, by the group $\mathrm{Aut}(\mathbb{F}_q)$ of all field automorphisms of the field $\mathbb{F}_q$ of q elements. For $n \geq 4$ and $n \neq 6$, there exist two not isomorphic groups G occurring in a short exact sequence $1 \to C_2 \to G \to S_n \to 1$ where G does not contain a subgroup isomorphic to A_n. For $n = 4$, one of the groups is isomorphic to $GL(2,3)$ and the other, denoted here by $\hat{S}_4$, has exactly one element of order 2. Henceforth, for a finite group G, $F(G)$ denotes the Fitting subgroup of G, i.e., the largest nilpotent normal subgroup of G, and $F^2(G)$ denotes the pre-image of $F(G/F(G))$ under the quotient map $G \to G/F(G)$. Moreover, for two groups N and H, $N \rtimes H$ denotes a semi-direct product of N and H, i.e., the splitting extension G associated with a short exact sequence $1 \to N \to G \to H \to 1$.

Classification Theorem. *Let G be a finite Oliver group with the rank integer $r_G \leq 1$. Then one of the following conclusions holds:*

(1) $G \cong PSL(2,q)$ *for some* $q \in \{5,7,8,9,11,13,17\}$*; or*
(2) $G \cong PSL(3,3)$, $PSL(3,4)$, $Sz(8)$, $Sz(32)$, A_7, M_{11} *or* M_{22}*; or*
(3) $G \cong PGL(2,5)$, $PGL(2,7)$, $P\Sigma L(2,8)$, *or* M_{10}*; or*
(4) G *is an extension of* $PSL(3,4)$ *by a unitary automorphism of order* 2*; or*
(5) $F(G) \cong C_2^2 \times C_3$ *and* G *is isomorphic either to* $\mathrm{Stab}_{A_7}(\{1,2,3\})$ *or to* $C_2^2 \rtimes D_9$*; or*
(6) $F(G)$ *is an abelian* p*-group for some odd prime* p, $G \cong F(G) \rtimes H$ *for* $H < G$ *with* $H \cong SL(2,3)$ *or* $\hat{S}_4$, *and* $F(G)$ *is inverted by the unique involution of* H*; or*
(7) $F(G) \cong C_3^3$ *and* $G \cong F(G) \rtimes A_4$*; or*
(8) $F(G) \cong C_2^4$, $F^2(G) \cong A_4 \times A_4$, *and* $G \cong F^2(G) \rtimes C_4$*; or*
(9) $F(G) \cong C_2^8$ *and* $G \cong F(G) \rtimes H$ *for* $H < G$ *with* $H \cong PSU(3,2)$ *or* $C_3^2 \rtimes C_8$*; or*
(10) $F(G) \cong C_2^3$ *and* $G/F(G) \cong GL(3,2)$*; or*
(11) $F(G) \cong C_2^4$ *and* $G/F(G) \cong A_6$*; or*
(12) $F(G) \cong C_2^8$ *and* $G/F(G) \cong M_{10}$*; or*
(13) $F(G)$ *is a non-identity elementary abelian* 2*-group and* $G/F(G)$ *is isomorphic to one of the groups* $SL(2,4)$, $\Sigma L(2,4)$, $SL(2,8)$, $Sz(8)$ *or* $Sz(32)$.

We remark that the Classification Theorem extends a previous result of Bannuscher and Tiedt [BT] obtained for finite nonsolvable groups G whose elements all have prime power order (i.e., whose rank integer $r_G = 0$). Our proof is largely independent of their result, but we do invoke it to establish that $F(G)$ is elementary abelian in case (13). Also, their result and our cases (1)–(13) allow us to list all finite Oliver groups G with $r_G = 1$.

Let G be a finite group. We denote by $\mathcal{P}(G)$ the family of subgroups of G of prime power order. For a prime p, by the *Dress subgroup of G of type p* we mean the smallest normal subgroup G^p of G such that $|G/G^p| = p^a$ for an integer $a \geq 0$ (cf. [LM]).

A subgroup H of G is called a *large subgroup of* G if H contains the Dress subgroup G^p of G of type p for some prime p. We denote by $\mathcal{L}(G)$ the family of large subgroups of G. A real G-module V is called *$\mathcal{L}$-free* if $\dim V^H = 0$ for each large subgroup H of G.

Let $LO(G)$ be the subgroup of $RO(G)$ formed by taking the differences $U - V \in RO(G)$ represented by two real $\mathcal{L}$-free G-modules U and V which are isomorphic when restricted to any $P \in \mathcal{P}(G)$. Recall that $IO(G)$ is the intersection of the kernels of the restriction maps $RO(G) \to RO(P)$ taken for all $P \in \mathcal{P}(G)$. Also, $PO(G) = IO(G) \cap \operatorname{Ker}(RO(G) \to \mathbb{Z})$, where $RO(G) \to \mathbb{Z}$ is the G-fixed point set dimension map. Clearly, $LO(G) \subseteq PO(G)$. In general, $LO(G) \neq PO(G)$. However, $LO(G) = PO(G)$ if G is a finite perfect group, because then the Dress subgroup $G^p = G$ for any prime p, and thus $\mathcal{L}(G) = \{G\}$.

The following theorem generalizes the assertion of Corollary 1.8 in [LP], concerned with the realifications of complex G-modules for any finite perfect group G.

Realization Theorem. *Let G be a finite Oliver gap group. Then $LO(G) \subseteq Sm(G)$.*

In the Realization Theorem, we in fact prove that $LO(G) \subseteq Pm(G)$ because we always know that $LO(G) \subseteq PO(G)$ and $Pm(G) = PO(G) \cap Sm(G)$ by definition.

Here, as in [MSY], a finite group G is called a *gap group* if $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ and G has a real $\mathcal{L}$-free G-module V satisfying the gap condition that $\dim V^P > 2 \dim V^H$ for each pair (P, H) of subgroups P and H of G such that $P < H$ and $P \in \mathcal{P}(G)$.

We refer to [MSY] and [Sum] for basic information on gap groups. According to [DH] or [MSY], the symmetric group S_n is a gap group if and only if $n \geq 6$. It turns out that if G is a finite group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$, then G is a gap group assuming that:

(1) $G^p \neq G$ and $G^q \neq G$ for two distinct odd primes p and q (which is true, e.g., when G has a cyclic quotient of order pq for two distinct odd primes p and q), or
(2) $G^2 = G$ (which is true, e.g., when G is perfect or G is of odd order), or
(3) G has a quotient which is a gap group.

In Theorems A1–A3, B1–B3, and C1–C3 below, we wish to state further results about $LO(G)$, $Pm(G)$, and $Sm(G)$. In particular, we dedect cases where $LO(G) \neq 0$, and using the Realization Theorem, we describe finite Oliver gap groups G with $Pm(G) \neq 0$. Also, using representation theory, we determine classes of finite groups G with $Sm(G) = 0$.

Theorem A1. *Let G be a finite group with a cyclic quotient of order pq for two distinct odd primes p and q. Then $r_G \geq 2$ and $LO(G) \neq 0$.*

Theorem A2. *Let G be a finite Oliver group of odd order. Then $r_G \geq 2$ and $LO(G) \neq 0$.*

Theorem A3. *Let G be a finite nonsolvable group. Then*

(1) *$LO(G) = 0$ for $r_G \leq 1$,*
(2) *$LO(G) \neq 0$ for $r_G \geq 2$, except when $G \cong \mathrm{Aut}(A_6)$ or $P\Sigma L(2,27)$, and*
(3) *$LO(G) = 0$ and $r_G = 2$ when $G \cong \mathrm{Aut}(A_6)$ or $P\Sigma L(2,27)$.*

Theorem B1. *Let G be a finite Oliver group with a cyclic quotient of order pq for two distinct odd primes p and q. Then G is a gap group, $r_G \geq 2$, and $Pm(G) \neq 0$.*

Theorem B2. *Let G be a finite Oliver group of odd order. Then G is a gap group, $r_G \geq 2$, and $Pm(G) \neq 0$.*

Theorem B3. *Let G be a finite nonsolvable gap group not isomorphic to $P\Sigma L(2,27)$. Then $Pm(G) \neq 0$ if and only if $r_G \geq 2$.*

If G is a finite perfect group, $LO(G) = PO(G) \subseteq Sm(G)$ by the Realization Theorem, and thus $Pm(G) = PO(G)$. Hence, $Pm(G) \neq 0$ if and only if $r_G \geq 2$, proving Theorem B3 in the case G is a finite perfect group. Note that in this case, the assertion of Theorem B3 is equivalent to the assertion of Theorem A in [LP].

Conjecture 2 holds for any group G occurring in Theorems B1 and B2. By Theorem B3, Conjecture 2 also holds for any finite nonsolvable gap group G with $r_G \geq 2$, except when $G \cong P\Sigma L(2,27)$. In this exceptional case, $r_G = 2$ and $LO(G) = 0$ by Theorem A3, and the same is true when $G \cong \mathrm{Aut}(A_6)$. According to [MSY, Proposition 4.1], $P\Sigma L(2,27)$ is a gap group while $\mathrm{Aut}(A_6)$ is not. In the case $G \cong P\Sigma L(2,27)$ or $\mathrm{Aut}(A_6)$, $r_G = 2$ and thus $\mathrm{rk}\, PO(G) = 1$, but we do not know whether $PO(G) \subseteq Sm(G)$.

By Theorem B3, $Sm(G) \neq 0$ for any finite nonabelian simple group G with $r_G \geq 2$. Using the Classification Theorem, we will determine all finite nonabelian simple groups G with $r_G \leq 1$, and in each case, we will check that $Sm(G) = 0$. If $G \cong \mathbb{Z}_p$ for some prime p, then $r_G = 0$ and $Sm(G) = 0$ by [AB], [Mi]. So, for any finite simple group G, the answer to the Smith isomorphism question reads as follows: $Sm(G) = 0$ if and only if $r_G \leq 1$.

We wish to recall that A_n is a simple group if and only if $n \geq 5$. Moreover, except for $PSL(2,2)$ and $PSL(2,3)$, every $PSL(n,q)$ is a simple group, and the following holds: $A_5 \cong PSL(2,4) \cong PSL(2,5)$, $A_6 \cong PSL(2,9)$, and $PSL(2,7) \cong PSL(3,2)$.

Theorem C1. *Let G be a finite nonabelian simple group. Then the following holds.*

(1) *For $r_G \leq 1$, $Sm(G) = 0$ and G is isomorphic to one of the groups:*
$r_G = 0:$ $PSL(2,q)$ *for* $q = 5, 7, 8, 9, 17$, *or* $PSL(3,4)$, $Sz(8)$, $Sz(32)$,
$r_G = 1:$ $PSL(2,11)$, $PSL(2,13)$, $PSL(3,3)$, A_7, M_{11}, M_{22}.
In particular, G has a cyclic subgroup of order 8 if and only if G is isomorphic to $PSL(2,17)$, $PSL(3,3)$, M_{11}, or M_{22}.

(2) *For $r_G \geq 2$, $Pm(G) \neq 0$ and thus $Sm(G) \neq 0$.*

Theorem C2. *Let $G = SL(n,q)$ or $Sp(n,q)$ for $n \geq 2$ where n is even in the latter case and q is any prime power in both cases. Then the following holds.*

(1) *For $r_G \leq 1$, $Sm(G) = 0$ and G is isomorphic to one of the groups:*
$r_G = 0:$ $SL(2,2)$, $SL(2,4)$, $SL(2,8)$, $SL(3,2)$,
$r_G = 1:$ $SL(2,3)$, $SL(3,3)$.
In particular, G has a cyclic subgroup of order 8 if and only if $G = SL(3,3)$.

(2) *For $r_G \geq 2$, $Pm(G) \neq 0$ and thus $Sm(G) \neq 0$.*

Theorem C3. *Let $G = A_n$ or S_n for $n \geq 2$. Then the following holds.*

(1) *For $r_G \leq 1$, $Sm(G) = 0$ and G is one of the groups:*
$r_G = 0:$ A_2, A_3, A_4, A_5, A_6, S_2, S_3, S_4,
$r_G = 1:$ A_7, S_5.
In particular, G has no cyclic subgroup of order 8.

(2) *For $r_G \geq 2$, $Pm(G) \neq 0$ and thus $Sm(G) \neq 0$.*

We recall that the symplectic group $Sp(n,q)$ as well as the projective symplectic group $PSp(n,q)$ are defined for any even integer $n \geq 2$ and any prime power q.

Concluding Corollary. *The statement $Sm(G) = 0$ if and only if $r_G \leq 1$ holds for any finite simple group G and for $G = SL(n,q)$, $PSL(n,q)$, $Sp(n,q)$, $PSp(n,q)$, A_n, or S_n, where $n \geq 2$ and q is any prime power, and n is even for $G = Sp(n,q)$ or $PSp(n,q)$.*

We wish to note that in the Concluding Corollary, for $G = A_n$, the result follows from [LP, Theorem B]. Moreover, in [LP, Theorem B], the results about $PSL(n,q)$ and $SL(n,q)$ are obtained for $n = 2$ and any prime q. While [LP] deals with the realifications of complex G-modules, in our paper (more generally), we deal with real G-modules.

The material of this paper is organized as follows. In Section 0, for any finite group G and any normal subgroup H of G, we define a subgroup $IO(G,H)$ of $PO(G)$ and we compute the rank of the free abelian group $IO(G,H)$ to the effect that $\operatorname{rk} IO(G,H) = r_G - \overline{r}_{G/H}$, where $\overline{r}_{G/H}$ is the number of the real conjugacy classes $(gH)^{\pm 1}$ of G/H represented by cosets gH containing elements of G not of prime power order (Lemma 0.1). Moreover,

$$IO(G,G^{\text{sol}}) \subseteq IO(G,G^{\text{nil}}) \subseteq LO(G) \subseteq IO(G,G^{p}) \subseteq IO(G,G) = PO(G)$$

for any Dress subgroup G^p of G and the smallest normal subgroups G^{sol} and G^{nil} of G such that G/G^{sol} is solvable and G/G^{nil} is nilpotent (Lemma 0.3). In Section 1, we obtain our first major result about the rank interger r_G (Proposition 1.6). The result asserts that for a finite Oliver group G of odd order and without cyclic quotient of order pq for two distinct odd primes p and q, $r_G > \overline{r}_{G/G^{\text{nil}}}$ and thus $IO(G,G^{\text{nil}}) \neq 0$ by Lemma 0.1.

In Section 2, we prove the Classification Theorem by using the fundamental results of [GLS] including those restated in Theorems 2.2–2.4 of this paper. Moreover, we make use of Burnside's $p^a q^b$ Theorem, the Feit-Thompson Theorem, the Brauer-Suzuki Theorem, and the Classification of the finite simple groups. In Section 3, by arguing as we do in Section 2 for a finite nonsolvable group G, we analyze the cases $r_G = \overline{r}_{G/G^{\text{sol}}}$ and in effect, we obtain our next major result about the rank interger r_G (Proposition 3.1). In particular, we dedect the cases where $r_G > \overline{r}_{G/G^{\text{sol}}}$ and thus $IO(G,G^{\text{sol}}) \neq 0$ by Lemma 0.1.

In Section 4, for any finite Oliver group G, we introduce the notion of Oliver equivalence of G-modules to the effect that in the notion of Smith equivalence, we replace actions on spheres by actions on disks. Then, using [O3, Theorem 0.4], we show that any two Oliver equivalent real G-modules are isomorphic if and only if G is listed in the cases (1)–(13) of the Classification Theorem (Corollary 4.3). Moreover, we describe real $\mathcal{L}$-free G-modules which can be realized as the tangent G-modules at isolated fixed points for smooth actions of G on disks (Theorem 4.4). In Section 5, we prove the Realization Theorem by using Theorem 4.4 and a similar theorem obtained for actions on spheres (Theorem 5.5) which in turn we prove by applying an equivariant surgery result (Theorem 5.2).

In Section 6, we consider a class of finite groups G satisfying the 8-condition, which implies that $Sm(G) \subseteq PO(G)$, and thus $Sm(G) = 0$ when in addition $r_G \leq 1$.

In Section 7, we prove Theorems A1–A3, B1–B2, C1–C3, and the Concluding Corollary. We refer to [Br], [tD], or [K] for background material on transformation groups.

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0. Rank Lemma

Let G be a finite group. We already defined the following series of free abelian groups: $LO(G) \subseteq PO(G) \subseteq IO(G)$. Now, for a normal subgroup H of G, we put

$$IO(G,H) = IO(G) \cap \operatorname{Ker}(RO(G) \xrightarrow{\operatorname{Fix}^H} RO(G/H))$$

with Fix^H defined by setting $\operatorname{Fix}^H(U-V) = U^H - V^H$ for two real G-modules U and V, where the H-fixed point sets U^H and V^H are considered as the canonical G/H-modules. If $U - V \in IO(G,H)$, then $U - V \in IO(G)$ and $U^H \cong V^H$ as G/H-modules, and thus

$$\dim U^G = \dim (U^H)^{G/H} = \dim (V^H)^{G/H} = \dim V^G,$$

proving that $U - V \in PO(G)$. So $IO(G,H) \subseteq PO(G)$. As $RO(G) \xrightarrow{\operatorname{Fix}^G} RO(G/G) \cong \mathbb{Z}$ is the G-fixed point set dimension map, $IO(G,G) = PO(G)$.

Recall that for a finite group G, the rank integer of G is the number r_G of the real conjugacy classes $(g)^{\pm 1}$ of G such that elements g of G are not of prime power order.

For a normal subgroup H of G, we denote by $\overline{r}_{G/H}$ the number of the real conjugacy classes $(gH)^{\pm 1}$ of G/H represented by cosets gH containing elements of G not of prime power order. In general, $r_G \geq \overline{r}_{G/H} \geq r_{G/H}$. Clearly, $r_G = \overline{r}_{G/G} = 0$ when each element of G has prime power order, and otherwise either $r_G = \overline{r}_{G/G} = 1$ or $r_G > \overline{r}_{G/G} = 1$. Thus $r_G = \overline{r}_{G/G}$ if and only if $r_G = 0$ or 1.

Now, we compute the rank of the free abelian group $IO(G,H)$. In the special case where $H = G$, the computation goes back to [LP, Lemma 2.1].

Lemma 0.1 (Rank Lemma). *For a finite group G and a normal subgroup H of G,*

$$\operatorname{rk} IO(G) = r_G, \quad \operatorname{rk} IO(G,H) = r_G - \overline{r}_{G/H}, \quad \textit{and} \quad \operatorname{rk} PO(G) = r_G - \overline{r}_{G/G}.$$

In particular, $IO(G,H) = 0$ for $r_G \leq 1$, and $PO(G) = 0$ if and only if $r_G \leq 1$.

Proof. In [LP, Lemma 2.1], the rank of $IO(G)$ is computed as follows. The rank of the free abelian group $IO(G)$ is equal to the dimension of the real vector space $\mathbb{R} \otimes_{\mathbb{Z}} IO(G)$ which consists of the real valued functions on G that are constant on the real conjugacy classes $(g)^{\pm 1}$ and that vanish when g is of prime power order. Therefore, $\operatorname{rk} IO(G) = r_G$. Now, for a normal subgroup H of G, we compute the rank of the kernel

$$IO(G,H) = \operatorname{Ker}(IO(G) \xrightarrow{\operatorname{Fix}^H} RO(G/H)).$$

First, for a representation $\rho : G \to \mathrm{GL}(V)$, consider the corresponding representation $\mathrm{Fix}^H \rho : G/H \to \mathrm{GL}(V^H)$ given by $(\mathrm{Fix}^H \rho)(gH) = \rho(g)|_{V^H}$ for each $g \in G$. Let $\pi : V \to V$ be the projection of V onto V^H, that is,

$$\pi = \frac{1}{|H|} \sum_{h \in H} \rho(h) : \ V \to V.$$

Then the trace of $(\mathrm{Fix}^H \rho)(gH) : V^H \to V^H$ is the same as the trace of the endomorphism

$$\rho(g) \circ \pi = \frac{1}{|H|} \sum_{h \in H} \rho(gh) : \ V \to V.$$

So, if χ is the character of ρ, then the character $\mathrm{Fix}^H \chi$ of $\mathrm{Fix}^H \rho$ is given by

$$(\mathrm{Fix}^H \chi)(gH) = \frac{1}{|H|} \sum_{h \in H} \chi(gh).$$

This formula extends (by linearity) to $\mathbb{R} \otimes_{\mathbb{Z}} RO(G)$. Consider the basis of $\mathbb{R} \otimes_{\mathbb{Z}} IO(G)$ consisting of the functions $f_{(g)^{\pm 1}}$ which have the value 1 on $(g)^{\pm 1}$ and 0 otherwise, defined for all classes $(g)^{\pm 1}$ represented by elements $g \in G$ not of prime power order. Then, by the formula above applied to $\chi = f_{(g)^{\pm 1}}$,

$$(\mathrm{Fix}^H f_{(g)^{\pm 1}})(gH) = \frac{|(g)^{\pm 1} \cap gH|}{|H|}$$

and $\mathrm{Fix}^H f_{(g)^{\pm 1}}$ vanishes outside of $(gH)^{\pm 1}$. Thus the map $\mathrm{Fix}^H : IO(G) \to RO(G/H)$ has image of rank $\overline{r}_{G/H}$ and its kernel $IO(G, H)$ is of rank $r_G - \overline{r}_{G/H}$. □

As already noted, $IO(G, H) \subseteq IO(G, G) = PO(G)$ for any normal subgroup H of G. More generally, the following lemma holds.

Lemma 0.2. *Let G be a finite group with two normal subgroups H and K of G such that $H \subseteq K$. Then $IO(G, H)$ is a subgroup of $IO(G, K)$.*

Proof. Take an element $U - V \in IO(G, H) = \mathrm{Ker}\,(IO(G) \xrightarrow{\mathrm{Fix}^H} RO(G/H))$ and consider the G-orthogonal complements $U - U^H$ and $V - V^H$ of the real G-modules U and V. Then $U - V = (U - U^H) - (V - V^H)$ because $U^H \cong V^H$ as G/H-modules. But

$$(U - U^H) - (V - V^H) \in IO(G, K) = \mathrm{Ker}\,(IO(G) \xrightarrow{\mathrm{Fix}^K} RO(G/K))$$

because $(U - U^H)^K = (V - V^H)^K = \{0\}$, proving that $U - V \in IO(G, K)$. □

We wish to recall the definitions of the characteristic subgroups G^{sol} and G^{nil} of a given finite group G, which were used in [LMP] and [LM], respectively.

By the definitions in [LMP], G^{sol} and G^{nil} are the smallest normal subgroups of G such that the quotient G/G^{sol} is a solvable group and the quotient G/G^{nil} is a nilpotent group. Clearly, G is a perfect group if and only if $G^{\mathrm{sol}} = G$, and G is a solvable group if and only if G^{sol} is the trivial group. Moreover, G is a nilpotent group if and only if G^{nil} is the trivial group. Note that $G^{\mathrm{nil}} = \bigcap G^p$ taken for all Dress subgroups G^p of G.

Lemma 0.3. *Let G be a finite group and let p be a prime. Then the following holds:*

$$IO(G, G^{\mathrm{sol}}) \subseteq IO(G, G^{\mathrm{nil}}) \subseteq LO(G) \subseteq IO(G, G^p) \subseteq IO(G, G) = PO(G).$$

Proof. As $G^{\mathrm{sol}} \subseteq G^{\mathrm{nil}}$, Lemma 0.2 asserts that $IO(G, G^{\mathrm{sol}}) \subseteq IO(G, G^{\mathrm{nil}})$. To prove that $IO(G, G^{\mathrm{nil}}) \subseteq LO(G)$, set $H = G^{\mathrm{nil}}$ and for a real G-module V, consider V^H as a real G-module with the canonical action of G. Note that the G-orthogonal complement $V - V^H$ of V^H in V is $\mathcal{L}$-free. Thus, for $U - V \in IO(G, H)$, it follows that

$$U - V = (U - U^H) - (V - V^H) \in LO(G).$$

Finally, $LO(G) \subseteq IO(G, G^p)$ because any element of $LO(G)$ is the difference of two real $\mathcal{L}$-free G-modules U and V with $U - V \in IO(G)$. In particular, $\dim U^{G^p} = \dim V^{G^p} = 0$ and $U - V \in IO(G)$ which implies that $U - V \in IO(G, G^p)$. □

Lemma 0.3 yields rank inequalities for the corresponding free abelian groups.

Corollary 0.4. *Let G be a finite group and let p be a prime. Then the following holds:*

(1) $\operatorname{rk} IO(G, G^{\mathrm{sol}}) \leq \operatorname{rk} IO(G, G^{\mathrm{nil}}) \leq \operatorname{rk} LO(G) \leq \operatorname{rk} IO(G, G^p) \leq \operatorname{rk} PO(G)$, *and*
(2) $\operatorname{rk} LO(G) \leq \min\{\operatorname{rk} IO(G, G^p) \mid p \big| |G|\}$.

Remark 0.5. Let G be a finite group and let H be a normal subgroup of G. If $H \neq G$, then one of the following conclusions holds:

(1) $r_G = \overline{r}_{G/H} = 0$ when each element of G has prime power order, and otherwise
(2) $r_G = \overline{r}_{G/H} = 1$ (holds, e.g., for $G = S_5$ and $H = G^{\mathrm{sol}} = A_5$), or
(3) $r_G = \overline{r}_{G/H} > 1$ (holds, e.g., for $G = \mathrm{Aut}(A_6)$ and $H = G^{\mathrm{sol}}$), or
(4) $r_G > \overline{r}_{G/H} = 1$ (holds, e.g., for $G = S_6$ and $H = G^{\mathrm{sol}} = A_6$), or
(5) $r_G > \overline{r}_{G/H} > 1$ (holds, e.g., for $G = A_5 \times \mathbb{Z}_3$ and $H = G^{\mathrm{sol}} = A_5$).

1. Finite Oliver groups of odd order

It will be convenient to use the following notation. For a finite group G, we denote by $\mathrm{NPP}(G)$ the set of all elements of G whose order is not a power of a prime, and we refer to the elements of $\mathrm{NPP}(G)$ as NPP elements of G. We also denote by $\overline{\mathrm{NPP}}(G)$ the set of real conjugacy classes which are subsets of $\mathrm{NPP}(G)$. Therefore, the rank integer r_G is the number of elements in $\overline{\mathrm{NPP}}(G)$.

For a finite group G and a normal subgroup H of G, it follows from Lemma 0.1 that $IO(G,H) \neq 0$ if and only if $r_G > \overline{r}_{G/H}$. Moreover, $r_G > \overline{r}_{G/H}$ if and only if $\mathrm{NPP}(G)$ contains two elements x and y which are not real conjugate in G but such that the cosets xH and yH are real conjugate in G/H. This allows us to prove the following lemma.

Lemma 1.1. *Let G be a finite group and let $H \trianglelefteq G$, i.e., H is a normal subgroup of G. Then the following three conclusions hold.*

(1) *Some coset gH meets two members of $\overline{\mathrm{NPP}}(G)$ if and only if $r_G > \overline{r}_{G/H}$.*
(2) *If H contains two distinct members of $\overline{\mathrm{NPP}}(G)$, then $r_G > \overline{r}_{G/H}$.*
(3) *If $r_G = \overline{r}_{G/H}$, then $r_{G/K} = \overline{r}_{(G/K)/(H/K)}$ for any $K \trianglelefteq G$ with $K \subseteq H$.*

Proof. The first conclusion is immediate from the remarks above and the second is a special case of the first. In order to prove the third conclusion, suppose that $r_{G/K} > \overline{r}_{(G/K)/(H/K)}$. Then some coset $\overline{g}(H/K)$ meets two members of $\overline{\mathrm{NPP}}(G/K)$. Assume that x and y are elements of G such that $\overline{x}$ and $\overline{y}$ are not of prime power order in G/K and are not in the same real conjugacy class of G/K. Then neither x nor y is of prime power order and $xH = yH$. If $axa^{-1} \in \{y, y^{-1}\}$ for an element $a \in G$, then $\overline{a}\,\overline{x}\,\overline{a}^{-1} \in \{\overline{y}, \overline{y}^{-1}\}$ contrary to assumption. Hence, x and y are not in the same real conjugacy class of G and so $r_G > \overline{r}_{G/H}$ by the first conclusion, proving the third one. □

Lemma 1.2. *Let G be a finite group and assume that $K \trianglelefteq L \subseteq H \subseteq G$ is a sequence of subgroups of G such that the quotient L/K contains NPP elements of two different orders. Then H contains NPP elements of two different orders. In particular, if $H \trianglelefteq G$, then $r_G > \overline{r}_{G/H} \geq r_{G/H}$.*

Proof. Suppose xK and yK are NPP elements of L/K of different orders. If x and y have different orders, we are done. If not, we may assume that the order of x is larger than the order of xK, in which case the cyclic group generated by x contains two NPP elements of different orders. So in any case, H contains two NPP elements of different orders, and if $H \trianglelefteq G$, then $r_G > \overline{r}_{G/H}$ by Lemma 1.1. Clearly $\overline{r}_{G/H} \geq r_{G/H}$. □

Lemma 1.3. *Let G be a finite nonsolvable group with a nonsolvable subgroup B and a cyclic subgroup $C \neq 1$ such that BC is a subgroup of G isomorphic to $B \times C$. Then G has NPP elements of different orders, and thus $r_G \geq 2$. If furthermore $B \subseteq H$ for $H = G^{\mathrm{sol}}$, then $r_G > \overline{r}_{G/H}$.*

Proof. For a prime divisor p of the order of C, choose an element $g \in C$ of order p. Since B is nonsolvable, it follows from Burnside's $p^a q^b$ Theorem that the order of B has (at least) three distinct prime divisors q, q', and q'', say with $p \neq q$ and $p \neq q'$. Choose two elements h and h' in B of orders q and q', respectively. By the assumption, BC is a subgroup of G (which amounts to saying that $BC = CB$) isomorphic to $B \times C$. Thus, the elements gh and gh' have orders pq and pq', respectively, proving that $r_G \geq 2$. If furthermore $B \subseteq H$ for $H = G^{\mathrm{sol}}$, then the coset gH contains the elements gh and gh' which are not real conjugate in G. Therefore, $r_G > \overline{r}_{G/H}$ by Lemma 1.1. □

Lemma 1.4. *Let G be a finite group of odd order. Let H be a normal subgroup of G. Suppose that p is a prime and P is an abelian p-subgroup of H with $P \trianglelefteq G$. Suppose that q is a prime, $q \neq p$, and $x \in H$ of order q with $V = C_P(x) \neq 1$. Then $r_G > \overline{r}_{G/H}$.*

Proof. Suppose that $r_G = \overline{r}_{G/H}$. Thus, by Lemma 1.1, H contains at most one member of $\overline{\mathrm{NPP}}(G)$. On the other hand, every element of $Vx \smallsetminus \{x\}$ has order pq and so all of these elements lie in the same member of $\overline{\mathrm{NPP}}(G)$. Let $h = yx$ and $h_1 = y_1x$ with y, $y_1 \in V \smallsetminus \{1\}$. Let $g \in G$ with $h^g = ghg^{-1} \in \{h_1, h_1^{-1}\}$. Then $y^g x^g \in \{y_1x, y_1^{-1}x^{-1}\}$ and so $x^g \in \{x, x^{-1}\}$. As $|G|$ is odd, $x^{-1} \notin x^G$. Hence $x^g = x$. Thus $g \in C_G(x)$. On the other hand, $C_G(x)$ normalizes $V = C_P(x)$ and so $C_G(x)$ transitively permutes the set $Vx \smallsetminus \{x\}$. But $|Vx \smallsetminus \{x\}| = |V| - 1$ is even, whereas $|C_G(x)|$ is odd, contradicting Lagrange's Theorem. □

Corollary 1.5. *Let G be a finite group of odd order. Let H be a normal subgroup of G such that $r_G = \overline{r}_{G/H}$. Then $F(H)$ is a p-group for some prime p and the Sylow q-subgroups of H are cyclic for all primes $q \neq p$.*

Proof. The result is trivial if $|H| = 1$. Otherwise let p be a prime divisor of $|F(H)|$ and let P be a nontrivial abelian normal subgroup of H. By Lemma 1.4, $C_P(x) = 1$ for all elements

x of H of prime order $q \neq p$. Thus $F(H)$ is a p-group. Moreover, if H contains a noncyclic abelian q-subgroup A for some prime $q \neq p$, then by Schur's Lemma (Theorem 2.3 below), $C_P(x) \neq 1$ for some element $x \in A$ of order q, a contradiction. Hence, as $|G|$ is odd, all Sylow q-subgroups of H are cyclic for $q \neq p$, as claimed. □

We can now obtain our first major result about the rank interger r_G.

Proposition 1.6. *Let G be a finite Oliver group of odd order. Let $H = G^{nil}$ and suppose that each cyclic quotient of G has prime power order. Then $r_G > \overline{r}_{G/H}$ and $r_G \geq 2$.*

Proof. If $r_G \leq 1$, then $r_G = \overline{r}_{G/H}$. Therefore we are done once we prove that $r_G > \overline{r}_{G/H}$. So, assume on the contrary that $r_G = \overline{r}_{G/H}$. Then Corollary 1.5 asserts that $F(H) = P$ is a p-group for some odd prime p. By assumption, for any prime $q \neq p$, G has no cyclic quotient of order pq. Hence G/H is an r-group for some prime r, and thus $H \neq F(H)$ because G is an Oliver group.

By Lemma 1.1 (3) applied for $K = P$, $r_{G/P} = \overline{r}_{(G/P)/(H/P)}$. Thus, by Corollary 1.5 applied to G/P, $F(H/P)$ is a q-group for some prime q. As $P = F(H)$, we have that $q \neq p$. Let F_2 be the pre-image in H of $F(H/P)$ and write $F_2 = PQ$ with Q a q-group. Again, by Corollary 1.5, Q is cyclic. Moreover, by Lemma 1.4, $C_P(Q) = 1$ and so $N = N_G(Q)$ is a complement to P in G by the Frattini argument. As Q is cyclic, $\mathrm{Aut}(Q)$ is abelian and so $N/C_G(Q)$ is abelian. Hence $PC_G(Q)$ is a normal subgroup of G with abelian quotient, whence $H \leq PC_G(Q)$, since H is the smallest normal subgroup of G with nilpotent quotient. But QP/P is the Fitting subgroup of H/P, whence $C_H(Q) \leq C_H(QP/P) = QP$. Thus $H = QP$. But then G is not an Oliver group, contrary to assumption. □

2. Finite Oliver groups G with $r_G = 0$ or 1

In this section, we shall classify finite Oliver groups G with the rank integer $r_G \leq 1$, proving the Classification Theorem stated in the introduction.

Theorem 2.1. *Let G be a finite Oliver group with the rank integer $r_G \leq 1$. Then one of the conclusions (1)–(13) in the Classification Theorem holds.*

In the proof of Theorem 2.1, our analysis will make repeated use of a few basic concepts and theorems. First, we recall that for a finite group H, the Fitting subgroup $F(H)$ of H is the largest normal nilpotent subgroup of H, $E(H)$ denotes the largest normal semisimple subgroup of H, and $F^*(H) = E(H)F(H)$ is the generalized Fitting subgroup of H, as defined by Helmut Bender. We shall use the fundamental results [GLS, Theorems 3.5, 3.6] describing the structure and embedding of $F^*(H)$.

Theorem 2.2 (Fitting-Bender Theorem). *For a finite group H, $[E(H), F(H)] = 1$ and $C_H(F^*(H)) = Z(F(H))$. If H is solvable, then $F^*(H) = F(H)$.*

Theorem 2.3 (Schur's Lemma). *If $E \cong C_p \times C_p$ acts on an abelian q-group V, where p and q are distinct primes, then $V = \langle C_V(e) : e \in E^{\#}\rangle$.*

Theorem 2.4. *If $F = K \rtimes H$ is a Frobenius group with kernel K and complement H and if F acts faithfully on a vector space V over a field of characteristic p, where $(p, |K|) = 1$, then $C_V(H) \neq 0$.*

We shall also use Burnside's $p^a q^b$ Theorem which asserts that the order of a finite nonsolvable group is always divisible by at least three distinct primes, the Feit-Thompson Theorem which asserts that finite groups of odd order are solvable; the Brauer-Suzuki Theorem which asserts that if G is a finite group with no nontrivial normal subgroup of odd order and with a Sylow 2-subgroup of 2-rank 1, then G has a unique involution z lying in $Z(G)$. Finally, we shall use the Classification of the finite simple groups.

The proof of Theorem 2.1 will be accomplished in a sequence of lemmas, the first two of which will address the following general situation: finite groups H without NPP elements; that is, all elements of H have orders that are powers of primes. Such groups are called *CP groups* and have been studied by several authors including Higman [H], Suzuki [Suz], Bannuscher–Tiedt [BT], and Delgado–Wu [DWu]. We remark that simple CP groups were first classified by Michio Suzuki in a deep paper [Suz], whose main theorem is one of the fundamental results in the proof of the classification of finite simple groups.

The following lemma goes back to Higman [H].

Lemma 2.5. *For a finite solvable CP group H, one of the following conclusions holds:*

(1) *H is a p-group for some prime p; or*
(2) *$H = K \rtimes C$ is a Frobenius group with kernel K a p-group and complement C a q-group of q-rank 1, for two distinct primes p and q; or*
(3) *$H = K \rtimes C \rtimes A$ is a 3-step group, in the sense that $K \rtimes C$ is a Frobenius group as in the conclusion (2) with C cyclic, and $C \rtimes A$ is a Frobenius group with kernel C and complement A a cyclic p-group.*

Corollary 2.6. *If G is a finite Oliver CP group, then $F^*(G)$ is nonsolvable.*

Proof. As none of the groups in the conclusions of Lemma 2.5 is an Oliver group, the result follows by Lemma 2.5. □

Now, we analyze the situation where a finite nonabelian simple group L is without NPP elements or all NPP elements of L have the same order.

Lemma 2.7. *Let L be a finite nonabelian simple group. Assume that L is without NPP elements or all NPP elements of L have the same order. Then L is isomorphic to one of the following groups:*

(1) $PSL(2,q)$ *with* $q \equiv \pm 3 \pmod 8$*; or*
(2) $PSL(2,q)$ *with* $q = 9$ *or* q *a Fermat or Mersenne prime; or*
(3) $PSL(2,2^n)$ *or* $Sz(2^n)$, $n \geq 3$*; or*
(4) $PSL(3,3)$, $PSL(3,4)$, A_7, M_{11} *or* M_{22}.

Proof. We survey the finite simple groups freely making use of the information in [GLS] and [C–W]. If $L \cong A_n$ for some $n \geq 8$, then L contains elements of orders 6 and 15, contrary to assumption. By inspection of [GLS, Tables 5.3], if L is a sporadic simple group, then $L \cong M_{11}$ or M_{22}.

Hence, we may assume that L is a finite simple group of Lie type defined over a field of characteristic p. Assume that L is not isomorphic to $PSL(2,q)$ or $Sz(2^n)$. Then by consideration of subsystem subgroups ([GLS, Section 2.6]), we see that one of the following statements is true about L:

(1) L contains a subgroup K with $K/Z(K) \cong PSL(4,p)$, $PSU(4,p)$, $PSp(6,p)$, $G_2(p)$ or ${}^2F_4(2)'$; or
(2) $L \cong PSL(3,q)$, $PSU(3,q)$ or $PSp(4,q)$.

Suppose first that p is odd. Then $PSp(4,p)$, $PSL(4,p)$, $PSU(4,p)$ and $G_2(p)$ all contain subgroups isomorphic to a commuting product of $SL(2,p)$ and a cyclic group of order 4. (See [GLS, Table 4.5.1].) Hence, each contains elements of order 6 and 12. Thus, we are reduced to the cases $L \cong PSL(3,q)$ or $L \cong PSU(3,q)$. In both cases, L contains a subgroup isomorphic to $SL(2,q)$. If $q > 3$, then $SL(2,q)$ contains an element of odd prime order $r > 3$ and hence elements of orders 6 and $2r$, contrary to assumption. Thus, we may assume that $L \cong PSL(3,3)$ or $PSU(3,3)$. We readily check that $PSU(3,3)$ contains elements of order 12 (cf. [GLS] or [C–W]), completing the case when p is odd.

Now suppose that $p = 2$. Now $SL(3, 2^n)$ contains a subgroup isomorphic to $GL(2, 2^n)$, whence $PSL(3, 2^n)$ contains $H = J \times C$, where $J \cong SL(2, 2^n)$ and C is cyclic of order $2^n - 1$ or $\frac{2^n-1}{3}$. If $n > 2$, this contradicts Lemma 1.7. Similarly, $SU(3, 2^n)$ contains a subgroup isomorphic to $GU(2, 2^n)$, whence $PSU(3, 2^n)$ contains $H_1 = J_1 \times C_1$ with $J_1 \cong SL(2, 2^n)$ and C_1 cyclic of order $2^n + 1$ or $\frac{2^n+1}{3}$. If $n > 1$, this again contradicts Lemma 1.7. Finally, $PSp(4, 2^n) = Sp(4, 2^n)$ contains a subgroup isomorphic to $GL(2, 2^n)$, again giving a contradiction when $n > 1$.

Now, $PSL(4, 2) \cong A_8$, $PSU(4, 2) \cong PSp(4, 3)$ and $G_2(2)' \cong U_3(3)$. By inspection in [C–W], $Sp(6, 2)$ and ${}^2F_4(2)'$ have elements of orders 6 and 10. We conclude that the only examples with $p = 2$ are $PSL(3, 2) \cong PSL(2, 7)$, $PSL(3, 4)$ and $Sp(4, 2)' \cong A_6$.

Finally, suppose that $L \cong PSL(2, q)$ and $q \cong \varepsilon \pmod 8$, $\varepsilon = \pm 1$. Then L has a cyclic subgroup of order $\frac{q-\varepsilon}{2}$. If r is an odd prime divisor of $q - \varepsilon$, then L has elements of order $2r$ and $4r$, contrary to assumption. Hence, q is a Fermat or Mersenne prime, or $q = 9$, completing the proof. □

Lemma 2.8. *Suppose that $F^*(G) = L$ is a finite nonabelian simple group and $r_G = \overline{r}_{G/L}$. Then G is isomorphic to one of the following groups:*

(1) *$PSL(2, q)$, $q \in \{5, 7, 8, 9, 11, 13, 17\}$; or*
(2) *$Sz(8)$, $Sz(32)$, A_7, $PSL(3, 3)$, $PSL(3, 4)$, M_{11} or M_{22}; or*
(3) *$PGL(2, 5)$, $PGL(2, 7)$, $P\Sigma L(2, 8)$, M_{10}, $\mathrm{Aut}(A_6)$, $P\Sigma L(2, 27)$ or the extension $PSL(3, 4)^*$ of $PSL(3, 4)$ by an involutory graph-field automorphism of order 2.*

If G is a CP group, then G is isomorphic to one of the following groups: $PSL(2, q)$, $q \in \{5, 7, 8, 9, 17\}$; or $Sz(8)$, $Sz(32)$, $PSL(3, 4)$ or M_{10}. Moreover, if $G = \mathrm{Aut}(A_6)$ or $P\Sigma L(2, 27)$, then $r_G = 2$. In all other cases, $r_G \le 1$.

Proof. Certainly L is one of the groups listed in Lemma 2.7. First, suppose that $G \neq L$. Note that the hypotheses imply that for any $x \in G$, all NPP elements of the coset Lx have the same order. By easy inspection (or use of [C–W]), if

$$L \in \{PSL(3, 3), PSL(3, 4), A_7, M_{11}, M_{22}\},$$

then $L \cong PSL(3, 4)$ and G is as described.

Suppose next that $L \cong Sz(2^n)$ and let $x \in G \smallsetminus L$ of prime order p. Then x induces a field automorphism on L and p divides n, whence p is odd. But $C_L(x)$ contains a subgroup $H \cong Sz(2)$, which contains a cyclic subgroup of order 4. Hence G contains elements of orders $2p$ and $4p$, contrary to assumption.

Finally suppose that $L \cong PSL(2, p^n)$. If $x \in G \smallsetminus L$ has prime order r and induces a field automorphism on L, then $C_L(x)$ contains a subgroup $H \cong PSL(2, p)$. If $p > 3$, then Lx contains elements of orders $2r$, $3r$ and pr, at least two of which are not prime powers, a contradiction. Hence $p \in \{2, 3\}$ and Lx contains elements of orders $2r$ and $3r$, whence $r \in \{2, 3\}$ and $C_L(x)$ is a $\{2, 3\}$-group. Hence $C_L(x) \cong PSL(2, 2)$, $PSL(2, 3)$ or $PGL(2, 3)$. Thus $p^n \in \{4, 8, 9, 27\}$. Now by inspection, we get the cases listed in Lemma 2.8.

Next suppose that $L \cong PSL(2, q)$, $q > 9$, and G has no element inducing a non-trivial field automorphism on L. As $L \neq G$, it follows that q is odd. Then by Lemma 2.7, q is an odd power of a prime and so $G \cong PGL(2, q)$. Let $q \cong \varepsilon \pmod 4$, $\varepsilon = \pm 1$. Then $G \smallsetminus L$ has an element x of order $q + \varepsilon$, and two elements y, y' in $\langle x \rangle$ are G-conjugate if and only if $y' = y^{-1}$. However Lx contains $\varphi(q + \varepsilon)$ elements of order $q + \varepsilon$, whence $\varphi(q + \varepsilon) = 2$ and $q + \varepsilon = 6$, contrary to the assumption that $q > 9$.

As $PSL(2, 4) \cong PSL(2, 5)$, we conclude that if $L \cong PSL(2, q)$, then $q \in \{5, 7, 8, 9, 27\}$, as claimed. The precise possibilities for G as stated in the proposition may then be inferred easily from [C–W].

Now suppose that $G = L \cong PSL(2, q)$. First we make a numerical remark. Suppose $2^n + 1 = 3^m$ for some natural numbers m and n. If m is odd, then $3^m - 1 \equiv 2 \pmod 8$ and so $m = 1$. If $m = 2r$, then $2^n = (3^r - 1)(3^r + 1)$ and so $m = 2$.

Now suppose that $G \cong PSL(2, q)$ with $q = 2^n > 8$. Then G has cyclic subgroups D_1 and D_2 of orders $2^n - 1$ and $2^n + 1$ respectively. If n is odd, then 3 divides $2^n + 1$, but $2^n + 1$ is not a power of 3 by the first paragraph. Hence $\mathrm{NPP}(G) \cap D_2$ contains $\varphi(2^n + 1)$ elements of order $2^n + 1$, lying in $\frac{\varphi(2^n+1)}{2}$ real classes. Hence, as $r_G \leq 1$, $\varphi(2^n + 1) \leq 2$, whence $2^n + 1 = 3$, a contradiction. Thus $n = 2s$ is even and 3 divides $2^n - 1 = (2^r - 1)(2^r + 1)$. As $n > 2$, D_1 is not a 3-group and, as above, $\frac{\varphi(2^n+1)}{2} \leq 1$. Then $2^n - 1 = 3$, again a contradiction.

Finally suppose that $G \cong PSL(2, q)$ with q odd and $q > 17$. Then G has cyclic subgroups T_1 and T_{-1} of orders $\frac{q-1}{2}$ and $\frac{q+1}{2}$ respectively. If $q \equiv \pm 3 \pmod 8$ and $q \equiv \varepsilon \pmod 4$, then $T_\varepsilon \cap \mathrm{NPP}(G)$ has $\varphi(\frac{q-\varepsilon}{2})$ elements of order $\frac{q-\varepsilon}{2}$ lying in $\frac{\varphi(\frac{q-\varepsilon}{2})}{2}$ real classes. Hence $\varphi(\frac{q-\varepsilon}{2}) = 2$, whence $\frac{q-\varepsilon}{4} = 3$, a contradiction.

Hence by Lemma 2.7, q is a Fermat or Mersenne prime. Again assume that $q \equiv \varepsilon \pmod 4$. As $q \neq 3$, 3 divides $q + \varepsilon$. Suppose that $q + \varepsilon = 2 \cdot 3^m$ for some $m \geq 2$. As $q - \varepsilon = 2^k$, we have $2^k + 2\varepsilon = 2 \cdot 3^m$. Hence $2^{k-1} = 3^m - \varepsilon$. If $\varepsilon = -1$, then $3^m + 1 \equiv 2$ or $4 \pmod 8$, whence $q \leq 9$, contrary to assumption. If $\varepsilon = 1$, then by the first paragraph,

$m \leq 2$ and $q \leq 17$, again a contradiction. It follows that $\frac{q+\varepsilon}{2}$ is not a prime power. But then $T_{-\varepsilon} \cap \mathrm{NPP}(G)$ has $\varphi(\frac{q+\varepsilon}{2})$ elements of order $\frac{q+\varepsilon}{2}$ lying in $\frac{\varphi(\frac{q+\varepsilon}{2})}{2}$ real classes. As usual this implies that $\varphi(\frac{q+\varepsilon}{2}) = 2$, again a contradiction.

Finally suppose that $G = L \cong Sz(2^n)$ with $n \geq 7$ and set $q = 2^n$. Then G has cyclic subgroups T_ε with $|T_\varepsilon| = q + \varepsilon\sqrt{2q} + 1$, for $\varepsilon = \pm 1$. As $q = 2^n$, n odd, we have that 5 divides $q^2 + 1 = |T_1||T_{-1}|$. Thus 5 divides $|T_\varepsilon|$ for some ε. We shall argue that $|T_\varepsilon|$ is not a power of 5 when $n \geq 7$. For suppose that it is. Let $n = 2m + 1$. Then

$$q + \varepsilon\sqrt{2q} + 1 = 2^{2m+1} + \varepsilon 2^{m+1} + 1 = 5^k$$

for some $k \geq 1$. Consideration of the 2-part of $5^k - 1$ shows that the smallest positive k for which $5^k - 1$ is divisible by 2^{m+1}, $m \geq 1$, is $k = 2^{m-1}$. But $2^{2m+1} + 2^{m+1} + 1 < 2^{2m+2}$, while $5^{2^{m-1}} > 4^{2^{m-1}} = 2^{2^m}$. Thus for equality to hold, we must have $2^m < 2m + 2$, which holds only for $m \leq 2$, i.e. for $n \leq 5$.

Thus for $G \cong Sz(2^n)$, $n \geq 7$, the cyclic subgroup T_ε is generated by elements in $\mathrm{NPP}(G)$. Let $h = |T_\varepsilon|$. As $N_G(T_\varepsilon)/T_\varepsilon$ has order 4 and $r_G = 1$, we must have $\varphi(h) = 4$, whence $h = 5$, a final contradiction. □

Henceforth, we shall assume that G is a finite Oliver group with the properties that

(1) all elements of $\mathrm{NPP}(G)$ have the same order; and
(2) if K is a normal subgroup of G such that $K \cap \mathrm{NPP}(G) \neq \varnothing$, then $\mathrm{NPP}(G) \subseteq K$.

We shall call G an *EP group* if the properties (1) and (2) above hold. Of course, both of these properties hold when $r_G \leq 1$. Notice that by Lemma 1.6, the class of EP groups is closed under taking subgroups and homomorphic images.

Lemma 2.9. *Suppose that G is an EP group and $F(G)$ is not a p-group. Then G is solvable and the following conclusions hold:*

(1) *$F(G) = P \times Q$ with P an elementary abelian p-group of order p^a and Q an elementary abelian q-group of order q^b; and either*
(2) *$G/F(G)$ is an r-group of r-rank 1, r prime; with $r \notin \{p, q\}$; or*
(3) *$G/F(G)$ is a nonabelian metacyclic Frobenius group of order $p^c q^d$.*

Proof. Clearly $\mathrm{NPP}(G) \subseteq Z(F(G))$, whence $F(G) = P \times Q$ with P and Q elementary abelian, as in (1). If $G = F(G)$, then G is not an Oliver group, contrary to assumption. Hence $G \neq F(G)$. Set $\overline{G} = G/F(G)$. If $\overline{G}$ has r-rank greater than 1 for some prime r, then from Schur's Lemma it follows that $G \smallsetminus F(G)$ contains elements of order rs for $s \in \{p,q\} \smallsetminus \{r\}$, a contradiction to $\mathrm{NPP}(G) \subseteq F(G)$. Hence $\overline{G}$ has r-rank 1 for every prime divisor r of $|\overline{G}|$.

Suppose that $F(\overline{G}) = 1$. Then by the Feit-Thompson Theorem, $\overline{G}$ has no nontrivial normal subgroup of odd order. Moreover $\overline{G}$ has 2-rank 1, whence by the Brauer-Suzuki Theorem, $1 \neq Z(\overline{G}) \leq F(\overline{G})$, a contradiction. Thus $F(\overline{G}) \neq 1$ and, as $\mathrm{NPP}(\overline{G}) = \varnothing$, $F(\overline{G}) = \overline{R}$ is an r-group of r-rank 1 for some prime r. Moreover note that $C_{\overline{G}}(Z(\overline{R}))$ is a normal r-subgroup of $\overline{G}$, whence $C_{\overline{G}}(Z(\overline{R})) = \overline{R}$. Moreover note that $\overline{G}/\overline{R}$ is isomorphic to a cyclic r'-subgroup of $\mathrm{Aut}(Z(\overline{R}))$.

If $\overline{G} = \overline{R}$, then as G is an Oliver group, $r \notin \{p,q\}$ and (2) holds. Otherwise $\overline{G} = \overline{R} \rtimes \overline{C}$ with both $\overline{R}$ and $\overline{C}$ cyclic. As in Lemma 2.5, C is an s-group for some prime $s \neq r$. Choose $s \in \{p,q\} \smallsetminus \{r\}$. As $\overline{R} \rtimes \overline{C}$ is a Frobenius group, $C_P(\overline{C}) \neq 1$ and so $s = p$. If also $r \neq q$, then the same argument would yield $s = q$, a contradiction. Hence $r = q$ and $s = p$ with $p < q$, whence (3) holds. □

Lemma 2.10. *Suppose that G is a finite Oliver group with $r_G \leq 1$ and with $F(G)$ not a p-group. Then $F(G) \cong C_2^2 \times C_3$ and one of the following holds:*

(1) *$G \cong \mathrm{Stab}_{A_7}(\{1,2,3\})$; or*
(2) *$G \cong C_2^2 \rtimes D_9$.*

Proof. We continue the notation of Lemma 2.9. Then $|\mathrm{NPP}(G)| = (p^a - 1)(q^b - 1)$ and $\mathrm{NPP}(G)$ is a union of one or two G-classes of equal cardinality.

If $|\overline{G}| = r^c$ for some prime r, and $R \in \mathrm{Syl}_r(G)$, then $C_R(P) = 1 = C_R(Q)$, whence $r^c \leq \min\{p^a - 1, q^b - 1\}$. On the other hand, $(p^a - 1)(q^b - 1) \leq 2r^c$, a contradiction.

Hence $|\overline{G}| = p^c q^d$ with $p < q$. Let $F(\overline{G}) = \overline{R}$ with $|\overline{R}| = q^d$, and let R be the full preimage of $\overline{R}$ in G. Then, as $F(G)$ is abelian, $1 \neq C_Q(R) \lhd G$. As all elements of $F(G)$ of order pq lie in the same real G-conjugacy class, this forces $C_Q(R) = Q$. Moreover $\overline{R}$ acts semi-regularly on $P^\#$, whence q^d divides $p^a - 1$. Let $\overline{S} \in \mathrm{Syl}_p(\overline{G})$. Then $\overline{S}$ acts semi-regularly on $Q^\#$, whence p^c divides $q^b - 1$. Finally $(p^a - 1)(q^b - 1) = p^c q^d$ or $2p^c q^d$. If both p and q are odd, then 4 divides the left-hand side of the equation but not the right. Hence $p = 2$. Then both $q^b - 1$ and $q^d + 1$ are powers of 2, whence $q = q^d = 3$ and $p^a = 4$. As $\overline{G}$ acts faithfully on P, it follows that $\overline{G} \cong S_3$. In particular $p^c = 2$, whence $q^b = 3$. Thus $F(G) \cong C_2^2 \times C_3$. Moreover, if $R_0 \in \mathrm{Syl}_3(G)$, then $G = P \rtimes N_G(R_0)$ and R_0 is inverted by an involution in $N_G(R_0)$. Thus either (1) or (2) holds. □

We keep our assumption that G is a finite Oliver group G, and we suppose that G is an EP group; i.e., all elements of $\mathrm{NPP}(G)$ have the same order, and if K is a normal subgroup of G such that $K \cap \mathrm{NPP}(G) \neq \varnothing$, then $\mathrm{NPP}(G) \subseteq K$.

Lemma 2.11. *Suppose that G is an EP group. Then one of the conclusions holds:*

1. *$F^*(G)$ is a p-group for some prime p; or*
2. *$F^*(G)$ is one of the nonabelian simple groups listed in Lemma 2.7; or*
3. *G is solvable and satisfies the conclusions of Lemma 2.9.*

Moreover if $r_G \leq 1$, then either $F^*(G)$ is a p-group or one of the conclusions of Lemma 2.8 or 2.10 holds.

Proof. Suppose that L is a normal quasisimple subgroup of $F^*(G)$. By Burnside's $p^a q^b$ Theorem, there exist distinct primes p, q and r dividing $|L|$. Thus if $C_{F^*(G)}(L) \neq 1$, then $\mathrm{NPP}(G)$ contains elements of two distinct orders, a contradiction. Hence $C_{F^*(G)}(L) = 1$, whence $L = F^*(G)$ is a nonabelian simple group and one of the conclusions of Lemma 2.7 (resp. 2.8) holds. On the other hand, if $F^*(G) = F(G)$, then either $F^*(G)$ is a p-group or one of the conclusions of Lemma 2.9 (resp. 2.10) holds, as claimed. □

Henceforth, we shall assume that $F^*(G) = P$ is a p-group. Clearly $G \neq P$ and we set $\overline{G} = G/P$. Also we let L be the full pre-image in G of $F^*(\overline{G})$.

Lemma 2.12. *Suppose that G is an EP group. Then either $\overline{L}$ is a q-group for some prime $q \neq p$ or $\overline{L}$ is a nonabelian simple group.*

Proof. Suppose not. Arguing as in Lemmas 1.5 and 1.6, we see that $\overline{G}$ is an EP group, provided that it is an Oliver group. If L is nonsolvable, then $\overline{G}$ is an Oliver group, whence Lemma 2.11 applied to $\overline{G}$ yields that $\overline{L}$ is a nonabelian simple group.

Suppose that L is solvable but $\overline{L}$ is not a q-group for any prime q. As all elements of $\mathrm{NPP}(G)$ have the same order, $\overline{L} = \overline{Q} \times \overline{R}$ where Q is an elementary abelian q-group and R is an elementary abelian r-group for some primes q, r different from p. Moreover G contains no elements of order pq or pr, whence $|Q| = q$ and $|R| = r$. As $\overline{L} = F^*(\overline{G})$ and $\overline{L}$ is cyclic of order qr, we conclude that G/L is abelian. As $\mathrm{NPP}(G) \subseteq L$, in fact G/L is abelian of prime power order. But then as P is a p-group and L/P is cyclic, G is not an Oliver group, contrary to assumption. □

Henceforth, we shall assume that G is a finite Oliver group with $r_G \leq 1$. In the next three lemmas, we shall treat the case where $\overline{L}$ is a q-group.

Lemma 2.13. *Suppose that $\overline{L}$ is a q-group of q-rank* 1*. Then $q = 2$ and $G = P \rtimes K$, where $K \cong SL(2,3)$ or $\hat{S}_4$ and P is an abelian p-group of odd order inverted by the unique involution of K.*

Proof. Suppose first that $\overline{L}$ is a cyclic q-group. As $\overline{L} = F^*(\overline{G})$, $\overline{G}/\overline{L}$ is a cyclic q'-group and $\overline{G}$ is a metacyclic Frobenius group with kernel $\overline{L}$. If $\overline{x} \in \overline{G} \smallsetminus \overline{L}$, then $C_P(\overline{x}) \neq 1$, whence G contains elements of order pr, if r is the order of x and $(r,p) = 1$. In this case $\overline{G}/\overline{L}$ has prime order r. Otherwise $\overline{G}/\overline{L}$ is a p-group. In either case, as $\overline{L}$ is cyclic, G is not an Oliver group, a contradiction.

Hence $\overline{L} \cong Q_8$ and $[\overline{G}, \overline{G}] \cong SL(2,3)$. As $\mathrm{NPP}(\overline{G}) \neq \varnothing$, P is inverted by z for any involution z of L. As $P = C_G(P)$, it follows that $G = P \rtimes K$ and K contains a unique involution, whence the lemma holds. □

Lemma 2.14. *Suppose that $\overline{L}$ is a q-group of q-rank greater than* 1*. Then $\overline{G}$ is a solvable group without NPP elements. Moreover, P is a finite elementary abelian p-group and $H = P \rtimes Q \rtimes C$, where $L = P \rtimes Q$, $Q \in \mathrm{Syl}_q(G)$ and $N_G(Q) = Q \rtimes C$ is a Frobenius group with kernel Q and complement C a p-group.*

Proof. Let $Z = \{z \in Z(P) : z^p = 1\}$. Then Z is a nontrivial elementary abelian normal p-subgroup of G. Let E be an elementary q-subgroup of L of q-rank greater than 1. By Schur's Lemma, $\mathrm{NPP}(L)$ contains an element x of order pq with $x^q \in Z$. Hence $\mathrm{NPP}(G/Z) = \varnothing$ and so, applying Schur's Lemma again in G/Z, we conclude that $P = Z$. If $L = G$, then G is not an Oliver group, contrary to assumption. Thus $L \neq G$.

Let $L = P \rtimes Q$, $Q \in \mathrm{Syl}_q(L)$. Suppose that $C_P(Q) = A \neq 1$. Then every element $x \in \mathrm{NPP}(G)$ satisfies $x^q \in A$. But then $C_P(e) = A$ for all $e \in E^\#$, whence $P = A$ by Schur's Lemma. But then $Q \leq C_G(P) = P$, a contradiction. Hence $C_P(Q) = 1$ and so $N_G(Q)$ is a complement to P in G, by a Frattini argument.

Let $N = N_G(Q)$. Then N is without NPP elements. Suppose that r is a prime divisor of $|N|$ with $r \notin \{p,q\}$ and let R be a nontrivial r-subgroup of N. As $\mathrm{NPP}(N) = \varnothing$, $Q \rtimes R$ is a Frobenius group with kernel Q acting faithfully on P. Hence $C_P(R) \neq 1$, a contradiction. Hence N is a $\{p,q\}$-group. In particular, N is solvable by Burnside's theorem and so Lemma 2.5 applies to N, yielding that either $N = Q \rtimes C$ is a Frobenius group with kernel Q and complement C a p-group, as claimed, or $N = Q \rtimes C \rtimes A$ with C a cyclic p-group and A a cyclic q-group disjoint from Q (as $CA = N_N(C)$ is a complement to Q in N). Suppose the latter and let $y \in A$ of order q and $z \in Q \cap Z(QA)$ of order q.

Then $U = \langle y, z\rangle \cong C_q \times C_q$ and so $P = \langle C_P(u) : u \in U^\#\rangle$ by Schur's Lemma. However $\mathrm{NPP}(G) \subseteq L$ and $U \cap L = \langle z\rangle$, whence $C_P(u) = 1$ for all $u \in U \smallsetminus \langle z\rangle$. Thus $P = C_P(z)$, a contradiction. □

Finally, we complete the analysis of the case when $\overline{L}$ is a q-group.

Lemma 2.15. *Suppose that $G = P \rtimes Q \rtimes C$ as in Lemma 2.14. Then one of the following conclusions hold:*

(1) *$P \cong C_3^3$ and $QC \cong A_4$; or*
(2) *$P \cong C_2^4$, $PQ \cong A_4 \times A_4$ and $C \cong C_4$; or*
(3) *$P \cong C_2^8$ and $QC \cong (C_3 \times C_3) \rtimes C_8$; or*
(4) *$P \cong C_2^8$ and $QC \cong (C_3 \times C_3) \rtimes Q_8$.*

Proof. Let $x \in Q$ of order q with $C_P(x) = V$ of maximum order. The elements of $V^\# x$ are in $\mathrm{NPP}(G)$. As $(vx)^{-1} \in Vx^{-1}$, either $C_G(x)$ is transitive on $V^\# x$ or $q = 2$ and $C_G(x)$ has two equal-sized orbits on $V^\# x$. In any case, as QC is a Frobenius group, $C_G(x) = VC_Q(x)$ and $V\langle x\rangle$ acts trivially on $V^\# x$. Hence $|V^\#| = |V^\# x| = q^b$ for some $b \geq 0$. Let $|V| = p^a$. Then either $p = 2$ and $b = 1$ or $q = 2$ and $p^a \in \{p, 9\}$. In all cases we set $Z = \{z \in Z(Q) : z^q = 1\}$. Thus Z is a normal elementary abelian q-subgroup of QC and $PZ \triangleleft G$.

Suppose first that $q = 2$. As ZC is a Frobenius group, Z contains a Klein 4-subgroup U and so $\mathrm{NPP}(G) \leq PZ \triangleleft G$. Thus in particular $x \in Z$. Suppose further that $|V| = p$. Then $|P| \leq p^3$. As P is a faithful QC-module, it follows that $|C| \leq dim V \leq 3$, whence $p = 3 = \dim V$ and $P \cong C_3^3$. As QC is isomorphic to a Frobenius $\{2, 3\}$-subgroup of $SL(3, 3)$, it follows that $QC \cong A_4$, and (1) holds.

Next suppose that $q = 2$ and $|V| = 9$. Note that $C_Q(v) \leq Z$ for all $v \in P^\#$. In particular $C_Q(V) = A$ is an elementary abelian q-group with $C_P(a) = V$ for all $a \in A^\#$, by maximal choice of V. If $A \neq \langle x\rangle$, then by Schur's Lemma, $P = V$, a contradiction. Hence $C_Q(V) = \langle x\rangle$ and $Q/\langle x\rangle$ is isomorphic to a subgroup of $GL(V) \cong GL(2, 3)$. In particular $|Q| \leq 32$. As C is fixed point free on Q, $|Q| = 2^m$, m even. As $|Q/Z| \leq 4$, $[Q, Q]$ is cyclic, whence $[Q, Q] = 1$ and $Q \cong C_2^2$, C_2^3 or $C_4 \times C_4$. On the other hand, $Q/\langle x\rangle$ is isomorphic to an abelian subgroup of $GL(2, 3)$ of order 8, whence $Q/\langle x\rangle \cong C_8$, a contradiction.

Finally suppose that $p = 2$ and $|V| = 2^a = q + 1$. As QC is a Frobenius group with C a 2-group, Q is abelian. Note that C permutes the set $\mathcal{Z} = \{z \in Z^\# : C_P(z) \neq 1\}$ in one or two equal-sized orbits. As $z \in \mathcal{Z}$ if and only if $\langle z \rangle^{\#} \leq \mathcal{Z}$, it follows that $|\mathcal{Z}| = k(q-1)$, $k \geq 1$. But also, as $|C|$ is a power of 2, $|\mathcal{Z}| = 2^c$ for some $c \geq 1$. Hence $q = 2^d + 1 = 2^a - 1$, whence $q = 3$ and $|V| = 4$. Now P is a completely reducible Z-module and as before $C_P(Z) = 1$, whence $P = P_1 \oplus \cdots \oplus P_r$ with P_i an irreducible Z-module and $|P_i| = 4$ for all i. Then $C_Z(P_1 \oplus P_2) = 1$, whence Q acts faithfully on $P_1 \oplus P_2 \cong C_2^4$. Hence $Q = Z \cong C_3 \times C_3$. As QC is a Frobenius group, either $C \cong Q_8$ or C is cyclic with $|C| \leq 8$. In any case the involution of C inverts Q. By Schur's Lemma, at least two cyclic subgroups of Q have non-trivial fixed points on P and, as $r_G = 1$, C permutes these subgroups transitively. Hence $|C| \geq 4$. If $|C| = 4$, then only two cyclic subgroups of Q have non-trivial fixed points on P and so $|P| = 16$ and $L \cong A_4 \times A_4$. Thus (2) holds.

If $|C| = 8$, then $\dim V \geq 8$. On the other hand, $\dim V \leq 8$ as Q has only four cyclic subgroups of order 3. Therefore, equality holds and G is as described either in (3) or (4), completing the proof. □

We have now completed the analysis of the case where $\overline{L}$ is a q-group. Thus, for the remainder of the analysis, we may assume that $\overline{L}$ is a nonabelian simple group.

Lemma 2.16. *Suppose that the p-group P is of odd order. Then P is elementary abelian and one of the following conclusions holds:*

(1) $p = 3$ *and* $\overline{G} \cong PSL(2, q)$, $q \in \{5, 7, 9, 17\}$, *or* $\overline{G} \cong M_{10}$*; or*
(2) $p = 7$ *and* $\overline{G} \cong SL(2, 8)$ *or* $Sz(8)$*; or*
(3) $p = 31$ *and* $\overline{G} \cong Sz(32)$.

Proof. Let $Z = \{z \in Z(P) : z^p = 1\}$. Then by Schur's Lemma, G contains an element x of order $2p$ with $x^2 \in Z$ and so every element of $\mathrm{NPP}(G)$ has this property. In particular $P = Z$ is elementary abelian, as usual. Moreover $\mathrm{NPP}(\overline{G}) = \varnothing$ and so $\overline{G} \cong PSL(2, q)$, $q \in \{5, 7, 8, 9, 17\}$, M_{10}, $Sz(8)$, $Sz(32)$ or $PSL(3, 4)$.

If $\overline{G}$ contains a subgroup isomorphic to $C_3 \times C_3$ or to A_4, then G contains elements of order $3p$ and so $p = 3$. Thus if $p > 3$, then $G \cong SL(2, 8)$, $Sz(8)$ or $Sz(32)$, as claimed. Moreover consideration of the Borel subgroups of $\overline{G}$ in these cases shows that $p = 7$, 7 or 31, respectively, as claimed.

Suppose finally that $p = 3$. If $\overline{G} \cong SL(2, 8)$, $Sz(8)$, $Sz(32)$ or $PSL(3, 4)$, then $\overline{L}$ contains a Frobenius group with kernel a 2-group and complement of order 7, 7, 31 or 5, respectively. But then G would contain elements of order 21, 21, 93 or 15, respectively, a contradiction. □

Lemma 2.17. *$F^*(G)$ is a 2-group.*

Proof. Suppose first that $p > 3$. Let $U \in \mathrm{Syl}_2(G)$. Then by consideration of the Frobenius group $\overline{B} = N_{\overline{G}}(\overline{U})$, we have $\dim(P) \geq p$. Hence if $V = C_P(z)$ for $z \in Z(U)^{\#}$, then $\dim(V) \geq \frac{1}{3}p > 2$. On the other hand $C_G(z)/V\langle z\rangle$ permutes $V^{\#}$ in at most two equal orbits. Hence $p^3 - 1 \leq |U|$, which is false in all cases.

Thus we may assume that $p = 3$ and $G \cong PSL(2,5)$, $PSL(2,7)$, $PSL(2,9)$, $PSL(2,17)$ or M_{10} and with $\dim(P) \geq 4$, 6, 4, 16, 8, respectively. Again let $U \in \mathrm{Syl}_2(G)$, $z \in Z(U)^{\#}$ and $V = C_P(z)$. Then $C_G(z) = VU$. Thus, as above, if $\dim(V) = d$, then $3^d - 1$ is a power of 2, whence $d \leq 2$ and so $\dim(P) \leq 3d \leq 6$. Hence $\dim(V) = 2$, $\dim(P) \leq 6$ and $|U| \geq 8$. Thus $\overline{G} \cong PSL(2,7)$ or $PSL(2,9)$. However, in both cases, $U/\langle z\rangle \cong C_2 \times C_2$, which cannot act semiregularly on $V^{\#}$ by Schur's Lemma, a contradiction. □

Lemma 2.18. *One of the following conclusions holds:*

(1) *$\overline{G} \cong PSL(2,q)$, $q \in \{5,7,8,9,17\}$; or*
(2) *$\overline{G} \cong Sz(8)$, $Sz(32)$, $PSL(3,4)$, $PGL(2,5)$ or M_{10}.*

Proof. If $\mathrm{NPP}(\overline{G}) = \varnothing$, then $\overline{G}$ is listed above. So we may assume that $\mathrm{NPP}(\overline{G}) \neq \varnothing$. Then G has no element x of odd order with $C_P(x) \neq 1$. In particular, by Schur's Lemma, G has a cyclic Sylow 3-subgroup, whence $\overline{G} \cong PGL(2,5)$, $PGL(2,7)$, $PSL(2,11)$ or $PSL(2,13)$. In the last three cases, $\overline{G}$ contains a Frobenius subgroup of order 21, 55 or 39, respectively, whence G contains elements x of order 3, 5, 3 respectively with $C_P(x) \neq 1$, a contradiction. Hence $\overline{G} \cong PGL(2,5)$. □

Lemma 2.19. *Suppose that $\overline{G} \in \{PSL(2,7), PSL(2,9), PSL(2,17), PSL(3,4), M_{10}\}$. Then one of the following conclusions holds:*

(1) *$P \cong C_2^3$ and $\overline{G} \cong GL(3,2)$; or*
(2) *$P \cong C_2^4$ and $\overline{G} \cong A_6 \cong Sp(4,2)'$; or*
(3) *$P \cong C_2^8$ and $\overline{G} \cong M_{10}$.*

Proof. Suppose that $\overline{G} \cong PSL(2,9)$, M_{10} or $PSL(3,4)$. Let $T \in \mathrm{Syl}_3(G)$. Then $T \cong C_3 \times C_3$ and $N_{\overline{G}}(\overline{T}) = \overline{T} \rtimes \overline{Q}$ with $\overline{Q} \cong C_4$, Q_8, Q_8, respectively, and with $\overline{T} \rtimes \overline{Q}$ a Frobenius group. Hence $\dim(P) \geq 4$, 8, 8, respectively. On the other hand, if $x \in T^{\#}$ and $V = C_P(x)$, then $C_G(x) = VT$ acts transitively on $V^{\#}$, whence $\dim(V) \leq 2$. As $\overline{Q}$ transitively permutes the set $\mathcal{T}$ of non-identity cyclic subgroups $\langle y\rangle$ of T with $C_P(y) \neq 1$, we have that $|\mathcal{T}| = 2$, 4,4, respectively. Hence $\dim(P) \leq 4$, 8, 8, respectively, whence equality holds in all cases. But then if $\overline{G} \cong PSL(3,4)$ and $g \in G$ of order 7, then $C_G(y) \neq 1$, whence G contains elements of order 6 and 14, a contradiction.

Next suppose that $\overline{G} \cong PSL(2,7)$. Let $x \in G$ of order 3. Then $\langle x \rangle$ is a Frobenius complement in a subgroup F of order 21, whence $C_P(x) \neq 1$. Thus G contains elements of order 6 and so G contains no elements of order 14. Thus P is a sum of faithful F-module, hence a sum of free $\langle x \rangle$-modules. On the other hand, as $C_G(x) = C_P(x)\langle x \rangle$, we must have $|C_P(x)| = 2$. Hence P is a single free $\langle x \rangle$-module, i.e. $|P| = 8$.

Finally suppose that $\overline{G} \cong PSL(2,17)$. By inspection of the 2-modular character table for $\overline{G}$, we see that if $x \in G$ of order 3, then $\dim(C_P(x)) \geq 3$. But $C_G(x) = C_P(x)X$ with $|X| = 9$, whence $C_G(x)$ is not transitive on $C_P(x)^{\#}$, a contradiction. □

Now, we can complete the proof of Theorem 2.1 as follows. The possibilities for $\overline{G}$ listed in Lemma 2.18 and not discussed in Lemma 2.19 are precisely those groups listed in the final conclusion of the Classification Theorem. For each of these cases, if $x \in G$ of odd order, then $\widetilde{C} = C_G(x)/\langle C_V(x), x \rangle$ must transitively permute $C_P(x)^{\#}$. However $|\widetilde{C}| \leq 2$, except in the cases when $\overline{G} = SL(2,8)$ or $Sz(32)$ and both x and $\widetilde{C}$ have order $p = 3$ or 5, respectively. Consideration of the 2-modular representations of these two groups shows that if W is an nontrivial irreducible 2-modular representation of $\overline{G}$ with $C_W(x) \neq 0$, then $|C_W(x)| > p+1$, and so $\widetilde{C}$ cannot act transitively on $C_P(x)^{\#}$. Thus in all of these cases, we must have $C_V(x) = 0$ for all $x \in G$ of odd order. Let $H = G^2$. Thus $H = G$ in all cases, except when $\overline{G} = \Sigma L(2,4) \cong S_5$. Then by the above remarks, H is a CP group and so by the theorem of Bannuscher-Tiedt [BT], the structure of H is as specified in the Classification Theorem, proving Theorem 2.1. □

The arguments of this section yield in particular the following corollary.

Corollary 2.20. *A finite nonabelian simple group G with the rank integer $r_G \leq 1$ is isomorphic to one of the following groups:*

$r_G = 0:$ $PSL(2,q)$ *for* $q = 5,7,8,9,17$*, or* $PSL(3,4)$, $Sz(8)$, $Sz(32)$,

$r_G = 1:$ $PSL(2,11)$, $PSL(2,13)$, $PSL(3,3)$, A_7, M_{11}, M_{22}.

3. Finite nonsolvable groups G with $r_G = \overline{r}_{G/G^{\mathrm{sol}}}$

In this section, we assume that G is a finite nonsolvable group, and we set $H = G^{\mathrm{sol}}$. We always have $r_G \geq \overline{r}_{G/H}$. We analyze the situation where $r_G = \overline{r}_{G/H}$, and this means that each coset gH meets at most one real conjugacy class $(x)^{\pm 1}$ with $x \in \mathrm{NPP}(G)$.

The main goal of this section is to prove the following proposition which contains our next major result about the rank interger r_G.

Proposition 3.1. *Let G be a finite nonsolvable group and let $H = G^{\mathrm{sol}}$. If $r_G = \overline{r}_{G/H}$, then either $r_G \leq 1$ or one of the following conclusions holds:*

(1) *$G = \mathrm{Aut}(A_6)$ and $r_G = \overline{r}_{G/H} = 2$; or*
(2) *$G = P\Sigma L(2, 27)$ and $r_G = \overline{r}_{G/H} = 2$.*

The proof will proceed via a sequence of lemmas. As the arguments are very similar to those in Section 2, we shall be a bit sketchy. We remark first that if $H = G$, then $\overline{r}_{G/H} \leq 1$ and there is nothing to prove. Hence we may assume that $H < G$. We let S denote the solvable radical of G, i.e. the largest normal solvable subgroup of G.

Lemma 3.2. *$S \leq H$ and $G/S \cong PGL(2, 5)$, $PGL(2, 7)$, $P\Sigma L(2, 8)$, M_{10}, $\mathrm{Aut}(A_6)$, $P\Sigma L(2, 27)$ or $PSL(3, 4)^*$.*

Proof. Let S_0 be the solvable radical of H and $\overline{G} = G/S_0$. As G is nonsolvable, $\overline{G}$ has a subnormal nonabelian simple subgroup $\overline{L} \leq \overline{H}$. By Lemmas 1.5(3) and 1.7, $C_{\overline{G}}(\overline{L}) = 1$. Hence $\overline{L} = F^*(\overline{G}) \neq \overline{G}$. Then the possibilities for $\overline{G}$ follow from Lemma 2.8. As $\overline{S} = 1$, $S_0 = S$ and the proof is complete. □

Lemma 3.3. *Either $S = 1$ or S is a p-group for some prime p.*

Proof. Suppose that $S \neq 1$. By Lemma 2.9, $F(G)$ is a p-group for some prime p. Let $\overline{G} = G/F(G)$ and $\overline{L} = F^*(\overline{G})$. Suppose that $\overline{L}$ is a nonabelian simple group. Then as G has only one nonabelian composition factor by Lemma 3.2, $H \leq L$, where L is the pre-image of $\overline{L}$ in G. Then $S = F(G)$ is a p-group, as claimed.

Thus by Lemma 2.12, we may assume that $\overline{L}$ is a q-group for some prime $q \neq p$. As $C_{\overline{G}}(\overline{L}) \leq \overline{L}$, it follows that $\mathrm{Aut}(\overline{L})$ is nonsolvable, whence $\overline{L}$ has q-rank at least 2. Thus L contains elements of order pq and so every NPP element of H lies in L. Since either p or q is odd, it follows that H/L has 2-rank 1. But then by the Brauer-Suzuki Theorem, H/L contains NPP elements, whence so does $H \smallsetminus L$, a contradiction. □

If $S = 1$, the conclusions of the theorem may be readily verified. Thus, it remains to prove that $S = 1$. We assume the contrary and argue to a contradiction. We set

$$\overline{G} = G/S \quad \text{and} \quad \overline{H} = H/S.$$

Lemma 3.4. *S is either a 2-group or an elementary abelian p-group for some odd prime p. If S is a p-group with p odd, then every NPP element of H has order $2p$.*

Proof. As $\overline{H}$ contains a Klein 4-group, the usual argument yields the result. □

Lemma 3.5. *$\overline{H}$ is not isomorphic to $PSL(2,27)$.*

Proof. Suppose that $\overline{H} \cong PSL(2,27)$. Then H contains an element x of order 14 with $x^{14} \notin S$. Hence H has no NPP element y with $y^r \in S$ for $r \in \{2,3\}$. However as $\overline{H}$ has 2-rank 2 and 3-rank 3, this contradicts Schur's Lemma for $r \neq p$. □

Lemma 3.6. *S is a 2-group and $\overline{G}$ is not isomorphic to M_{10}.*

Proof. Suppose first that S is a 2-group and $\overline{G} \cong M_{10}$. Then every element of $G \smallsetminus H$ is a 2-element, whence $\overline{r}_{G/H} \leq 1$, contrary to hypothesis.

Thus it remains to prove that S is a 2-group. Suppose not. Then S is an elementary abelian p-group for some odd prime p. Suppose that there is an involution $x \in G \smallsetminus H$. Then by inspection x centralizes a coset Hy of odd order, whence Hx contains NPP elements outside Sx. But then Sx contains no NPP elements, i.e. x inverts S and $H = [H,x]$ centralizes S, a contradiction. Thus there is no involution in $G \smallsetminus H$, and therefore we have the following two possibilities: $\overline{G} \cong P\Sigma L(2,8)$ or M_{10}.

Suppose that $\overline{G} \cong P\Sigma L(2,8)$. As $\overline{H}$ contains a Frobenius group of order 56, S must be a 7-group and H must contain elements of order 14. Thus a 3-element of H acts without fixed points on S. But then by Schur's Lemma, some element $x \in G \smallsetminus H$ of order 3 must have fixed points on S and so Hx contains elements of orders 6 and 21, a contradiction.

Finally suppose that $\overline{G} \cong M_{10}$. Then $\overline{H}$ contains an A_4-subgroup and threfore S is a 3-group and the NPP elements of H have order 6. Let t be an involution of H. As H contains only one real G-class of NPP elements, $C_G(t)$ must permute transitively the nonidentity elements of $C_S(t)$. Now $|C_G(t)/C_S(t)| = 16$ and $C_S(t)$ acts trivially on itself by conjugation. Hence $|C_S(t)| \leq 9$. Moreover, as H contains elements of order 6, H contains no elements of order 15. Hence, an element of H of order 5 acts fixed point freely on S, whence $\dim(S)$ is a multiple of 4. By Schur's Lemma, on the other hand, $\dim(S) \leq 3\dim(C_S(t))$, whence $|C_S(t)| = 9$. Let T be a Sylow 2-subgroup of G with $t \in Z(T)$. As t acts trivially on $C_S(t)$, $T/\langle t\rangle$ must act regularly on the eight elements of $C_S(t) \smallsetminus \{1\}$. But $T/\langle t\rangle$ is a dihedral group of order 8 and hence has no such regular action, a final contradiction. □

In all the lemmas below (Lemmas 3.7–3.11), x will be an element of H of order 3 with $U = C_S(x)$ and with $|U| = 2^a > 1$, if possible. We set $\widetilde{C} = C_G(x)/\langle U, x\rangle$.

Lemma 3.7. *U is elementary abelian and $\widetilde{C}$ transitively permutes the set $Ux \setminus \{x\}$ of cardinality $2^a - 1$. Moreover no chief H-factor in S is a trivial $\overline{H}$-module.*

Proof. As $|H/S|$ is divisible by at least two odd primes, but H has NPP elements of order $2p$ for at most one odd prime p, no chief H-factor in S is a trivial $\overline{H}$-module.

If $U = 1$, the lemma holds trivially. Suppose $U \neq 1$. As all NPP elements of H have order 6, all elements of U have order 2 and so U is elementary abelian. Moreover all elements of $Ux \setminus \{x\}$ are G-conjugate, hence $C_G(x)$-conjugate and as $\langle U, x\rangle$ is contained in the kernel of the conjugation action on Ux, the lemma follows. □

Lemma 3.8. *$\overline{G}$ is not isomorphic to $PGL(2,5)$.*

Proof. Suppose $\overline{G} \cong PGL(2,5)$. As $\overline{r}_{G/H} = 2$, H must contain NPP elements. By Lemma 3.7, we see that every chief H-factor of S is isomorphic to a 4-dimensional irreducible H/S-module. Thus if $y \in H$ of order 5, we have $C_S(y) = 1$. Hence H must have elements of order 6 and so with x and U as above, $U \neq 1$. Indeed some chief H-factor of S, say V, is a permutation module for $\overline{H}$ with $|C_V(x)| = 4$. Thus $|U| \geq 4$. But $|\widetilde{C}| = 2$ and so $\widetilde{C}$ does not act transitively on $Ux \setminus \{x\}$, contrary to Lemma 3.7. □

Lemma 3.9. *$\overline{H}$ is not isomorphic to $PSL(2,7)$.*

Proof. Suppose that $\overline{H} \cong PSL(2,7)$. As $\widetilde{C}$ is a 2-group acting transitively on the involutions of U, $|U| \leq 2$. According to [GLS2; 2.8.10], the nontrivial irreducible $GL(3,2)$ modules are the standard 3-dimensional module V, its dual V^* and the Steinberg module, which is the nontrivial constituent of $V \otimes V^*$. As a 3-element of $GL(3,2)$ has 1-dimensional fixed point space on V and V^* and 2-dimensional fixed point space on the Steinberg module, it follows that S has a unique irreducible composition factor and this has dimension 3, i.e. $S \cong V$ or V^* as $\overline{H}$-module. But then as $C_G(S) = S$ and $\mathrm{Aut}(S) \cong H/S$, $G = H$, a contradiction. □

Lemma 3.10. *$\overline{H}$ is not isomorphic to $PSL(2,8)$.*

Proof. Suppose not and let $y \in G \setminus H$ be an element of order 3. Then yS lies in a complement of a Frobenius subgroup of G/S of order 21. Hence there exists $ty \in Hy$ of order 6 with $(ty)^3 \in S$. However there also exists $sy \in Hy$ of order 6 with $(sy)^3 \in H \setminus S$, a contradiction. □

Lemma 3.11. *$\overline{H}$ is not isomorphic to $PSL(2,9)$ or $PSL(3,4)$.*

Proof. Suppose not. Then either $\overline{G} \cong \mathrm{Aut}(A_6)$ or $\overline{H} \cong PSL(3,4)$ with $|G:H| = 2$. In either case, $|\widetilde{C}| = 6$. Again as $\widetilde{C}$ acts transitively on the involutions of U, we conclude that $|U| \leq 4$. Let E be a Sylow 3-subgroup of H. Then $E \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $N_G(E)$ transitively permutes the elements of E of order 3. Hence $|C_T(y)| \leq 4$ for all $y \in E \smallsetminus \{1\}$ and so, by Schur's Lemma, $|S| \leq 2^8$.

Suppose that $\overline{H} \cong PSL(3,4)$. Then $G \smallsetminus H$ contains an element γ such that $\overline{\gamma}$ has order 2 and centralizes $N_{\overline{H}}(\overline{E})$ and $N_{\overline{G}}(\overline{E}) = \overline{EQ} \times \langle \overline{\gamma} \rangle$, where Q is a quaternion group of order 8 transitively permuting the nonidentity elements of E. Now $\overline{EQ}$ acts faithfully on $C_S(\gamma)$ by Thompson's $A \times B$ Lemma [GLS1; 11.7]. However, by Clifford Theory, a faithful $\overline{EQ}$ module must have dimension at least 8. As $\dim S \leq 8$, this would force $C_S(\gamma) = S$, which is absurd. Hence $\overline{G} \cong \mathrm{Aut}(A_6)$. Again $N_{\overline{G}}(\overline{E})$ contains a subgroup $\overline{EQ}$ with $\overline{Q} \cong Q_8$, as above. Thus $\dim(S) = 8$, $C_S(E) = 1$ and S is a faithful irreducible $N_G(E)$-module. In particular E acts nontrivially on $U = C_T(x) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and so $UE \cong A_4 \times \mathbb{Z}_3$. Now $\overline{G}$ contains a subgroup isomorphic to S_6 and so, by inspection, $N_G(E)$ contains an involution t centralizing x such that $E\langle t \rangle \cong S_3 \times \mathbb{Z}_3$. Then $UE\langle t \rangle \cong S_4 \times \mathbb{Z}_3$ with $x \in Z(UE\langle t \rangle)$. But then the coset Ht contains elements of order 6 and 12, whence $r_G > \overline{r}_{G/H}$, contrary to assumption. □

Now, the proof of Proposition 3.1 can be completed immediately by the results above. In fact, having exhausted all possible structures for G/S, we conclude that $S = 1$ and therefore the proof of Proposition 3.1 is complete. □

Now, using the rank integer r_G, we are able to determine completely the cases where $IO(G, G^{\mathrm{sol}}) \neq 0$ for finite nonsolvable groups G.

Corollary 3.12. *For a finite nonsolvable group G, the following conclusions hold:*

(1) *$IO(G, G^{sol}) = 0$ for $r_G \leq 1$,*
(2) *$IO(G, G^{sol}) \neq 0$ for $r_G \geq 2$, except when $G \cong \mathrm{Aut}(A_6)$ or $P\Sigma L(2,27)$, and*
(3) *$IO(G, G^{sol}) = 0$ and $r_G = 2$ when $G \cong \mathrm{Aut}(A_6)$ or $P\Sigma L(2,27)$.*

Proof. Set $H = G^{\mathrm{sol}}$. If $r_G \leq 1$, then $r_G = \overline{r}_{G/H}$, and thus $IO(G,H) = 0$ by Lemma 1.3. If $r_G \geq 2$, then except when $G = \mathrm{Aut}(A_6)$ or $P\Sigma L(2,27)$, $r_G > \overline{r}_{G/H}$ by Proposition 3.1, and thus $IO(G,H) \neq 0$ by Lemma 1.3. For $G = \mathrm{Aut}(A_6)$ or $P\Sigma L(2,27)$, $r_G = \overline{r}_{G/H} = 2$ by Proposition 3.1, and thus $IO(G,H) = 0$ by Lemma 1.3. □

4. Oliver equivalence of G-modules

Let G be a finite group. According to [O1] or [O3], there exists a smooth action of G on a disk with exactly two fixed points if and only if G is an Oliver group.

Let G be a finite Oliver group. Similarly as in the case of Smith equivalence notion, two real G-modules U and V are called *Oliver equivalent* if there exists a smooth action of G on a disk D^n with exactly two fixed points x and y, such that as real G-modules, $T_x(D^n) \cong U$ and $T_y(D^n) \cong V$.

We denote by $Dk(G)$ the subset of $RO(G)$ which is formed by taking the differences $U - V$ represented by two Oliver equivalent real G-modules U and V.

As we point out in the proof, the following theorem goes back to [O3, Theorem 0.4].

Theorem 4.1. *Let G be a finite Oliver group. Then $PO(G) \subseteq Dk(G)$.*

Proof. Take an element $U-V \in PO(G)$. Then the real G-modules U and V are isomorphic as P-modules for each $P \in \mathcal{P}(G)$, and by subtracting the trivial summands U^G and V^G from U and V, respectively, we may assume that $\dim U = \dim V = 0$. Now, according to [O3, Theorem 0.4], there exists a smooth action of G on a disk with exactly two fixed points at which the tangent G-modules are isomorphic to $U \oplus W$ and $V \oplus W$ for some real G-module W with $\dim W^G = 0$. Therefore $U - V = (U \oplus W) - (V \oplus W) \in Dk(G)$, proving that $PO(G) \subseteq Dk(G)$. □

Corollary 4.2. *Let G be a finite Oliver group. Then $Dk(G) = PO(G)$.*

Proof. The inclusion $Dk(G) \subseteq PO(G)$ holds by Smith theory and the Slice Theorem, and thus $Dk(G) = PO(G)$ by Theorem 4.1. □

Corollary 4.3. *For a finite Oliver group G, any two Oliver equivalent real G-modules are isomorphic if and only if G is listed in the Classification Theorem, (1)–(13).*

Proof. According to Lemma 0.1 and Corollary 4.2, $Dk(G) = 0$ if and only if $r_G \leq 1$, and in (1)–(13), the Classification Theorem lists all finite Oliver groups G with $r_G \leq 1$. □

Let G be a finite group. Following [LM], consider the real G-module

$$V(G) = (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_{p \,|\, |G|} (\mathbb{R}[G]^{G^p} - \mathbb{R})$$

where $\mathbb{R}[G]$ is the real regular G-module, the G^p-fixed point set $\mathbb{R}[G]^{G^p}$ has the canonical action of G, and the subtracted summands $\mathbb{R}$ all have the trivial actions of G.

The real G-module $V(G)$ has a number of useful properties that we use for constructing of smooth G-actions on disks and spheres. Here, we just recall that if G is a finite Oliver group, then $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ and $V(G)$ is $\mathcal{L}$-free (cf. [LM]).

In the proof of the Realization Theorem, we use the following theorem which allows us to obtain Oliver equivalent real $\mathcal{L}$-free G-modules.

Theorem 4.4. *Let G be a finite Oliver group. For an integer $k \geq 1$, let $V_1, \dots, V_k$ be real $\mathcal{L}$-free G-modules of dimension $d \geq 0$, such that $V_i - V_j \in IO(G)$ for all $1 \leq i, j \leq k$. Then, for a sufficiently large integer ℓ, there exists a smooth action of G on the n-disk D^n for $n = d + \ell \dim V(G)$, with exactly k fixed points at which the tangent G-modules are isomorphic to $V_1 \oplus \ell V(G), \dots, V_k \oplus \ell V(G)$.*

Proof. The result follows by arguing as in [MP2, the proof of Theorem 0.3] in the case where G is an Oliver group. □

5. Smith equivalence of G-modules

In this section, we prove the Realization Theorem stated in the introduction.

Theorem 5.1. *Let G be a finite Oliver gap group. Then $LO(G) \subseteq Sm(G)$.*

The proof of Theorem 5.1 follows from Theorem 4.4 and a number of results which we collect below. The key results are obtained in Theorem 5.5 and Corollary 5.9.

First, we recall that by using the equivariant surgery developed in [BM], [LMP], [LM], and [Mo1]–[Mo3], so called "deleting–inserting" theorems are proved in [LMP, Theorem 2.2] for a finite nonsolvable group G, and in [Mo3, Theorems 0.1 and 4.1] for a finite Oliver group G. These theorems allow us to modify a smooth action of G on a sphere S^n or a disk D^n with fixed point set F, in such a way that the resulting smooth action of G on S^n or D^n has a new fixed point set obtained from F by deleting or inserting a number of connected components of F. Here, we restate only the "deleting part" of [Mo3] for action on spheres in a modified form presented in [MP3, Theorem 3.1], where the G-orientation condition of [Mo3] is replaced by the $\mathcal{P}$-orientation condition of [MP3].

Let G be a finite group. A real G-module V is called *G-oriented* if V^H is oriented for each subgroup H of G, and the transformations $g : V^H \to V^H$ preserve orientation for all $g \in N_G(P)$. A real G-module V is called *$\mathcal{P}$-oriented* if V^P is oriented for each $P \in \mathcal{P}(G)$, and the transformations $g : V^P \to V^P$ preserve orientation for all $g \in N_G(P)$.

If a real G-module V is the realification of a complex G-module, then V is G-oriented. In particular, for any real G-module V, the G-module $V \oplus V$ is $\mathcal{P}$-oriented as it is the realification of the complexification of V.

Let M be a smooth manifold with the trivial action of G. A real G-vector bundle ν over M is called *$\mathcal{P}$-oriented* if each fiber of ν is $\mathcal{P}$-oriented as a real G-module. Similarly, a real G-vector bundle ν over M is called *$\mathcal{L}$-free* if each fiber of ν is $\mathcal{L}$-free as a real G-module.

For a G-space X, denote by $\mathcal{F}_{\text{iso}}(G; X)$ the family of isotropy subgroups G_x of G that occur at points $x \in X$. Given a subgroup H of G, set $X^{=H} = \{x \in X \mid G_x = H\}$.

For a finite group G, denote by $\mathcal{PC}(G)$ the family of subgroups H of G with a normal subgroup $P \in \mathcal{P}(G)$ such that H/P is cyclic. If G is a finite Oliver group, then by [LM], $\mathcal{PC}(G) \cap \mathcal{L}(G) = \varnothing$ and thus $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$.

Theorem 5.2. ([Mo3], [MP3]) *Let G be a finite Oliver group. Let F be the disjoint union of two closed smooth manifolds X and Y. Assume that there exists a smooth action of G on a homotopy n-sphere Σ^n such that $(\Sigma^n)^G = F$ and the following four conditions hold.*

1. *The equivariant normal bundle $\nu_{F \subset \Sigma^n}$ is $\mathcal{P}$-oriented and $\mathcal{L}$-free.*
2. *For each $P \in \mathcal{P}(G)$, $(\Sigma^n)^P$ is simply connected.*
3. *For each $P \in \mathcal{P}(G)$ and $H \in \mathcal{PC}(G)$, $\dim(\Sigma^n)^P \geq 5$ and $\dim(\Sigma^n)^{=H} \geq 2$.*
4. *For each $P \in \mathcal{P}(G)$ and $H \in \mathcal{S}(G)$ with $P < H$, $\dim(\Sigma^n)^P > 2\dim(\Sigma^n)^H$.*

Then there exists a smooth action of G on the standard n-sphere S^n such that $(S^n)^G = X$ and $\nu_{X \subset S^n} \cong \nu_{X \subset \Sigma^n}$ as G-vector bundles. Moreover, for each $P \in \mathcal{P}(G)$, $(S^n)^P$ is simply connected and $\dim(S^n)^P = \dim(\Sigma^n)^P$.

For two subgroups P and H of G, the pair (P, H) is called *proper* if $P \in \mathcal{P}(G)$ and P is a proper subgroup of H. Moreover, a proper pair (P, H) of subgroups of G is called *odd* if $|H : P| = |HG^2 : PG^2| = 2$ and $PG^p = G$ for all odd primes p. Finally, a proper pair (P, H) of subgroups of G is called *even* if (P, H) is not odd.

As in [MSY], for a real G-module V and a proper pair (P, H) of subgroups of G, we set $d_V(P, H) = \dim V^P - 2\dim V^H$. By [LM, Theorem 2.3], the following lemma holds.

Lemma 5.3. *Let G be a finite group. For a proper pair (P, H) of subgroups of G,*

1. *$d_{V(G)}(P, H) = 0$ when (P, H) is odd, and*
2. *$d_{V(G)}(P, H) > 0$ when (P, H) is even.*

A real G-module V is called a *gap* G-module if $d_V(P, H) > 0$ for each proper pair

(P, H) of subgroups of G. Recall that by definition, a finite group G is a gap group if $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ and G has a real $\mathcal{L}$-free gap G-module.

As noted above, if G is a finite Oliver group, then $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ and the real G-module $V(G)$ is $\mathcal{L}$-free. If in addition $G^p \neq G$ and $G^q \neq G$ for two distinct odd primes p and q, or $G = G^2$, then any proper pair (P, H) of subgroups of G is even (cf. [MSY]), and thus $V(G)$ is a gap G-module by Lemma 5.3, and so G is a gap group.

Now, we show that any finite gap group G has a real $\mathcal{L}$-free gap G-module which in addition is G-oriented (cf. the appendix) and contains $V(G)$ as a direct summand.

Lemma 5.4. *Let G be a finite gap group and let V be a real $\mathcal{L}$-free gap G-module. Then $2V \oplus 2V(G)$ is a real G-oriented (and thus $\mathcal{P}$-oriented) $\mathcal{L}$-free gap G-module.*

Proof. Clearly, the G-module $2V \oplus 2V(G)$ is both G-oriented and $\mathcal{L}$-free. Moreover, for each proper pair (P, H) of subgroups of G, $d_V(P, H) > 0$ as V is a gap G-module, and $d_{V(G)}(P, H) \geq 0$ by Lemma 5.3. So, for each proper pair (P, H) of subgroups of G,

$$d_{2V \oplus 2V(G)}(P, H) = 2d_V(P, H) + 2d_{V(G)}(P, H) > 0,$$

proving that $2V \oplus 2V(G)$ is a gap G-module. □

Now, using Theorem 5.2, we obtain a result for actions on spheres similar to that one obtained in Theorem 4.4 for actions on disks.

Theorem 5.5. *Let G be a finite Oliver gap group. Let V be a real $\mathcal{P}$-oriented $\mathcal{L}$-free gap G-module containing $V(G)$ as a direct summand. For an integer $k \geq 1$, let $V_1, \ldots, V_k$ be real $\mathcal{P}$-oriented $\mathcal{L}$-free G-modules of dimension $d \geq 0$, such that $V_i - V_j \in IO(G)$ for all $1 \leq i, j \leq k$. Then, for a sufficiently large interger ℓ, there exists a smooth action of G on the n-sphere S^n for $n = d + \ell \dim V$, with exactly k fixed points at which the tangent G-modules are isomorphic to $V_1 \oplus \ell V, \ldots, V_k \oplus \ell V$.*

Proof. According to [LM], $\mathcal{F}_{\text{iso}}(G; V(G) \smallsetminus \{0\}) = \mathcal{S}(G) \smallsetminus \mathcal{L}(G)$ and $\mathcal{PC}(G) \cap \mathcal{L}(G) = \varnothing$. Therefore, for each $H \in \mathcal{PC}(G)$, $\dim V(G)^{=H} \geq 1$ and thus $\dim V^{=H} \geq 1$.

For each $i = 1, \ldots, k$, consider the invariant unit sphere $\Sigma_i^n = S(V_i \oplus \ell V \oplus \mathbb{R})$, where G acts trivially on $\mathbb{R}$. In particular, $n = d + \ell \dim V$ for $d = \dim V_i$, and the fixed point set $F_i = (\Sigma_i^n)^G$ consists of two points, say x_i and y_i. Set $X_i = \{x_i\}$ and $Y_i = \{y_i\}$.

We claim that the conditions (1)–(4) of Theorem 5.2 all hold for Σ_i^n when ℓ is sufficiently large. First, note that the equivariant normal bundle $\nu_{F_i \subset \Sigma_i^n}$ consists of two copies of

$V_i \oplus \ell V$. As $V_i \oplus \ell V$ is $\mathcal{P}$-oriented and $\mathcal{L}$-free, (1) is proven. If $\ell \geq 2$, then for each $P \in \mathcal{P}(G)$, $\dim(\Sigma_i^n)^P \geq 2$ and thus the sphere $(\Sigma_i^n)^P$ is simply connected, proving (2). If $\ell \geq 5$, then for each $H \in \mathcal{PC}(G)$, $\dim(\ell V)^{=H} \geq 5$ which implies that $\dim(\Sigma_i^n)^{=H} \geq 5$, proving (3). Now, choose ℓ in such a way that $\ell\, d_V(P,H) > -d_{V_i}(P,H)$ for each proper pair (P,H) of subgroups of G (remember, as V is a gap G-module, $d_V(P,H) > 0$). Then, for each proper pair (P,H) of subgroups of G,

$$d_{V_i \oplus \ell V}(P,H) = d_{V_i}(P,H) + \ell\, d_V(P,H) > d_{V_i}(P,H) - d_{V_i}(P,H) = 0,$$

and thus $\dim(\Sigma_i^n)^P > 2\dim(\Sigma_i^n)^H$, proving (4). So, the claim is proven.

Now, by Theorem 5.2, we obtain a smooth action of G on a copy S_i^n of the n-sphere S^n such that $(S_i^n)^G = X_i = \{x_i\}$ and $T_{x_i}(S_i^n) \cong V_i \oplus \ell V$ as G-modules.

As $V_i - V_j \in IO(G)$ for $1 \leq i,j \leq k$, it follows from Theorem 4.4 that if the integer ℓ is sufficiently large, then there exists a smooth action of G on the n-disk D^n such that $(D^n)^G = \{a_1,\ldots,a_k\}$ and $T_{a_i}(D^n) \cong V_i \oplus \ell V$ for $i = 1,\ldots,k$. Clearly, the equivariant double of D^n is a copy S_0^n of the n-sphere S^n equipped with a smooth action of G such that $(S_0^n)^G = \{a_1,b_1,\ldots,a_k,b_k\}$ and as real G-modules:

$$T_{a_i}(S_0^n) \cong T_{b_i}(S_0^n) \cong V_i \oplus \ell V \cong T_{x_i}(S_i^n)$$

for $i = 1,\ldots,k$. This allows us to take the equivariant connected sum $S_0^n \# S_1^n \# \ldots \# S_k^n$ of the spheres $S_0^n, S_1^n, \ldots, S_k^n$ formed by connecting sufficiently small neighborhoods of points a_i and x_i. As a result, we obtain a smooth action of G on S^n such that $(S^n)^G = \{b_1,\ldots,b_k\}$ and $T_{b_i}(S^n) \cong V_i \oplus \ell V$ for $i = 1,\ldots,k$. $\square$

We wish to remark that using the methods of [Mo3], [Pa2], [O3], [MP1] and [MP2], one may prove more general results than that presented in Theorem 5.5. For example, in [MP3, Theorem 6.3], each isolated fixed point is replaced by a simply connected manifold. Without referring to [MP3, Theorem 6.3], we decided to give the proof of Theorem 5.5 due to simplifications which follow in the case where the fixed point set is discrete.

Let G be a finite group. For any real G-module V, the G-module $V \oplus V$ is the realification of the complexification of V. In particular, $V \oplus V$ is $\mathcal{P}$-oriented (as it is G-oriented).

In Proposition 5.8 below, we show that for any two real G-modules U and V such that $U - V \in IO(G)$, the G-module $U \oplus V$ is $\mathcal{P}$-oriented. Using some deep results about group actions on disks, the result is also noted in [MP3, Lemma 2.5]. Here, we make use only of group theory and representation theory arguments. First, we prove the following two lemmas about the determinants of transformations of G-modules.

Lemma 5.6. *Let G be a finite group and let $T = \langle t \rangle$ be the cyclic subgroup of G generated by an element $t \in G$ of 2-power order. Let U and V be two real G-modules of the same dimension. If $\dim U^T \equiv \dim V^T \pmod 2$, then the determinants of the transformations $t : U \to U$ and $t : V \to V$ agree, $\det(t|_U) = \det(t|_V)$.*

Proof. Let L be the nontrivial 1-dimensional real irreducible T-module determined by the canonical epimorphism $T \to O(1)$, which exists because the order of t is a power of 2. Then, as T-modules, $U \cong U^T \oplus mL$ and $V \cong V^T \oplus nL$ for some integers $m \geq 0$ and $n \geq 0$. As $\dim U = \dim V$ and $\dim U^T \equiv \dim V^T \pmod 2$, it follows that $m \equiv n \pmod 2$. Therefore, $\det(t|_U) = (-1)^m = (-1)^n = \det(t|_V)$. □

Lemma 5.7. *Let G be a finite group with a normal p-subgroup P for an odd prime p and with a cyclic 2-subgroup $T = \langle t \rangle$, such that $G = PT$. Let U and V be two non-zero real G-modules with $U^G = V^G = \{0\}$. If $U \cong V$ as P-modules, then the determinants of the transformations $t : U \to U$ and $t : V \to V$ agree, $\det(t|_U) = \det(t|_V)$.*

Proof. We proceed by induction on $|P| + \dim U$. If $P = 1$, then $U = V = \{0\}$, contrary to assumption. Let K be the kernel of the P-action on U (and V). If $K \neq 1$, we are done by induction in G/K. Hence we may assume that $K = 1$. Let E be a minimal normal subgroup of G with $E \leq P$. Suppose that $\dim U^E > 0$. Then $U^E \cong V^E$ and $U - U^E \cong V - V^E$ as P-modules and all four of these are G-modules. Hence induction yields that $\det(t|_{U^E}) = \det(t|_{V^E})$ and $\det(t|_{U-U^E}) = \det(t|_{V-V^E})$, and we are done.

Thus we may assume that $\dim U^E = \dim V^E = 0$. If $E \neq P$, we are done by induction in the group ET. Hence we may assume that P is an elementary abelian p-group and that P is a minimal normal subgroup of G. Also, $\dim U^P = \dim V^P = 0$. If $t' \in T$, then the centralizer $C_P(t')$ is normal in G, hence is 1 or P. Therefore either $G = P \times T$ is cyclic or the center $Z = Z(G)$ is a proper subgroup of T and the quotient G/Z is a Frobenius group with kernel PZ/Z and complement T/Z.

By assumption, $U \cong V$ as P-modules, and thus $\dim U = \dim V$. Hence, by Lemma 5.6, it suffices to prove that the congruence $\dim U^T \equiv \dim V^T \pmod 2$ holds.

Write $U = U_1 \oplus U_2 \oplus \dots$ and $V = V_1 \oplus V_2 \oplus \dots$ as sums of irreducible P-modules U_i and V_i with $U_i \cong V_i$. Then the inertia group of U_i in G equals the inertia group of V_i in G. Thus U and V are both the direct sums of the same number of irreducible G-modules and we may write $U = M_1 \oplus \cdots \oplus M_k$ and $V = N_1 \oplus \cdots \oplus N_k$ where M_i and N_i are G-modules with $\dim M_i = \dim N_i$ and $\dim M_i^P = \dim N_i^P = 0$.

So, in order to prove that the congruence $\dim U^T \equiv \dim V^T \pmod 2$ holds, it suffices to prove that for any two real irreducible G-modules M and N with $\dim M = \dim N$ and $\dim M^P = \dim N^P = 0$, the same congruence holds:

$$\dim M^T \equiv \dim N^T \pmod 2.$$

According to Clifford theory, if M is absolutely irreducible, $M = \mathrm{Ind}_{PZ}^G(W \otimes_{\mathbb{R}} \lambda)$ for an absolutely irreducible real P-module W and a character λ of $Z = Z(G)$. Set $d = |G : PZ|$. Then $\dim M = d$. On the other hand, if M is not absolutely irreducible, $M \otimes_{\mathbb{R}} \mathbb{C} = X \oplus \overline{X}$ where X and $\overline{X}$ are two complex conjugate absolutely irreducible G-modules, and so in this case $\dim M = 2d$. As $\dim M = \dim N$, either both M and N are absolutely irreducible or neither is.

Case 1: Neither M nor N is absolutely irreducible. Then $M \otimes_{\mathbb{R}} \mathbb{C} = X \oplus \overline{X}$ for two complex conjugate absolutely irreducible G-modules X and $\overline{X}$, and $N \otimes_{\mathbb{R}} \mathbb{C} = Y \oplus \overline{Y}$ for two complex conjugate absolutely irreducible G-modules Y and $\overline{Y}$. Thus the integers $\dim M^T$ and $\dim N^T$ are both even, proving that $\dim M^T \equiv \dim N^T \pmod 2$.

Case 2: M and N are both absolutely irreducible. Then $Z = Z(G)$ maps into the group $\{I, -I\}$ of real scalars and so we may assume that $|Z| \leq 2$. Furthermore if $|Z| = 1$, then we can replace G with a larger group so that $|Z| = 2$ without loss. Thus we may assume that $Z = \langle z \rangle$ has order 2. Let χ_M and χ_N be the characters of M and N, respectively. We wish to compute the Frobenius-Schur indicators $\nu(\chi_M)$ and $\nu(\chi_N)$ of the characters χ_M and χ_N. First, we set $\chi = \chi_M$. By definition,

$$\nu(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

Note that $\chi = \mathrm{Ind}_{PZ}^G(\lambda)$ for some irreducible character λ of PZ with $\mathrm{Res}_P^{PZ}(\lambda) \neq 1_P$. Since PZ is a normal subgroup of G, thus $\chi(g) = 0$ for all $g \in G \smallsetminus PZ$. Hence in the displayed sum, all the terms are 0 except when $g^2 \in PZ$. Let $v \in T$ with $v^2 = z$. Then $g^2 \in PZ$ if and only if $g \in P\langle v \rangle$, which is a union of two cosets of PZ. Consider the squaring map on PZ. This is a two-to-one map of PZ onto P (if $x \in P$, then $x^2 = (xz)^2 \in P$). Since $P\langle v \rangle = PZ \cup PZv$, we have

$$\nu(\chi) = \frac{1}{|G|} \left(2 \sum_{g \in P} \chi(g) + \sum_{g \in PZv} \chi(g^2) \right).$$

Now $\frac{1}{|P|}\sum_{g\in P}\chi(g) = \langle \mathrm{Res}^G_P(\chi), 1_P\rangle$, the inner product of $\mathrm{Res}^G_P(\chi)$ and 1_P; i.e. it is the multiplicity of 1_P as a constituent of $\mathrm{Res}^G_P(\chi)$, which is exactly the dimension of M^P, which is 0 by assumption. Hence $2\sum_{g\in P}\chi(g) = 0$ and so

$$\nu(\chi) = \frac{1}{|G|}\sum_{g\in PZv}\chi(g^2).$$

Let $x \in P$. Then $vxv^{-1} = x^{-1}$. Also $v^2 = z$. Hence $vxvx = vxv^{-1}v^2x = x^{-1}v^2x = z$. Also $vxzvxz = vxvx = z$. So $g^2 = z$ for all $g \in PZv$. Thus

$$\nu(\chi) = \frac{|PZ|\chi(z)}{|G|} = \frac{\chi(z)}{\chi(1)}.$$

So $\nu(\chi) = 1$ (whence χ is a real character) if and only if $\chi(z) = \chi(1) = d$, i.e. z is in the kernel of $\chi = \chi_M$. We may apply the same arguments for $\chi = \chi_N$ to conclude that $\nu(\chi) = 1$ if and only if z is in the kernel of $\chi = \chi_N$. Therefore, for both M and N, we may assume that z is in the kernel and factor it out. The G-modules M and N when restricted to T are both isomorphic to the regular T-module. Thus $\dim M^T = \dim N^T$, proving that the congruence $\dim M^T \equiv \dim N^T \pmod 2$ holds also in this case. □

Proposition 5.8. *Let G be a finite group. Let U and V be two real G-modules such that $U - V \in IO(G)$. Then, for each $g \in N_G(P)$ with $P \in \mathcal{P}(G)$, the determinants of the transformations $g : U^P \to U^P$ and $g : V^P \to V^P$ agree,* $\det(g|_{U^P}) = \det(g|_{V^P})$*, and thus* $\det(g|_{(U\oplus V)^P}) = 1$*, that is, $U \oplus V$ is $\mathcal{P}$-oriented.*

Proof. Let $t \in G$ be an element of 2-power order. If $g = tx = xt$ for an element $x \in G$ of odd order, then $\det(x) = 1$, so $\det(g) = \det(t)$. Thus it suffices to prove the result for $g = t$. By induction on the order of G, we may assume that $G = PT$ for a normal p-subgroup P of G and the cyclic subgroup T of G generated by t. If $p = 2$, G is a 2-group. Then $U \cong V$ as G-modules because $U - V \in IO(G)$, and the result is clear.

Assume that p is odd. As $U - V \in IO(G)$, $U \cong V$ both as P-modules and T-modules. Write $U = U^P \oplus (U - U^P)$ and $V = V^P \oplus (V - V^P)$. Then $\det(t|_{U^P}) = \det(t|_{V^P})$ if and only if $\det(t|_{U-U^P}) = \det(t|_{V-V^P})$. To complete the proof, note that the later equality holds by applying Lemma 5.7 to the action of G on the G-modules $U - U^P$ and $V - V^P$ (remember, $U - U^P$ and $V - V^P$ are isomorphic as P-modules). □

Finally, we are ready to present a result which allows us to realize each element of $LO(G)$ as the difference of two Smith equivalent real $\mathcal{L}$-free G-modules.

Corollary 5.9. *Let G be a finite Oliver gap group. For an integer $k \geq 1$, let $V_1, \ldots, V_k$ be real $\mathcal{L}$-free G-modules such that $V_i - V_j \in IO(G)$ for all $1 \leq i, j \leq k$. Then there exists a smooth action of G on a sphere S^n with exactly k fixed points at which the tangent G-modules are isomorphic to $V_1 \oplus W, \ldots, V_k \oplus W$ for a real $\mathcal{L}$-free G-module W such that $\dim W^P > 0$ for each $P \in \mathcal{P}(G)$.*

Proof. Let V_0 be one of the G-modules $V_1, \ldots, V_k$. Then each $V_i \oplus V_0$ is $\mathcal{L}$-free. Moreover, $V_i \oplus V_0$ is $\mathcal{P}$-oriented by Proposition 5.8. As G is a gap group, Lemma 5.4 asserts that G has a real $\mathcal{P}$-oriented $\mathcal{L}$-free gap G-module V which contains $V(G)$ as a direct summand. So, by Theorem 5.5, for a sufficiently large integer ℓ, there exists a smooth action of G on S^n with exactly k fixed points $x_1, \ldots, x_k$ and $T_{x_i}(S^n) \cong V_i \oplus W$ for $W = V_0 \oplus \ell V$ and all $1 \leq i \leq k$. Clearly, W is $\mathcal{L}$-free and since W contains $V(G)$ as a direct summand, $\dim W^P > 0$ for each $P \in \mathcal{P}(G)$. □

Proof of Theorem 5.1. Let G be a finite Oliver gap group. Take an element $U - V \in LO(G)$. As we may assume that the real G-modules U and V are both $\mathcal{L}$-free and $U - V \in IO(G)$, Corollary 5.9 asserts that there exists a smooth action of G on a sphere S^n with exactly two fixed points at which the tangent G-modules are isomorphic to $U \oplus W$ and $V \oplus W$ for a real $\mathcal{L}$-free G-module W. Therefore $U - V = (U \oplus W) - (V \oplus W) \in Sm(G)$, proving that $LO(G) \subseteq Sm(G)$. By Corollary 5.9, $\dim W^P > 0$ for each $P \in \mathcal{P}(G)$. □

Remark 5.10. In the proof of Theorem 5.1, we have shown that each element of $LO(G)$ can be realized as the difference of two Smith equivalent real $\mathcal{L}$-free G-modules $U \oplus W$ and $V \oplus W$ such that $\dim W^P > 0$ for each $P \in \mathcal{P}(G)$. In particular, by Smith theory, the action of G on the sphere S^n satisfies the condition that $(S^n)^P$ is connected for each $P \in \mathcal{P}(G)$, which in turn implies that the action of G on S^n is 2-proper so that $U \oplus W$ and $V \oplus W$ are 2-proper Smith equivalent in the sense of the definitions in [LP].

6. Smith equivalence and the 8-condition

In this section, we consider a class of finite groups G such that $Sm(G) \subseteq PO(G)$, and thus $Pm(G) = PO(G) \cap Sm(G) = Sm(G)$. For convenience, we say that a finite group G satisfies the 8-*condition* if G either has no element of order 8 or otherwise for any cyclic 2-subgroup H of G of order ≥ 8, $\dim W^H > 0$ for all irreducible G-modules W. In [LP], such a group G is called 2-*proper* (cf. [LP, Example 2.5 and Lemma 2.6]).

Lemma 6.1. *Let G be a finite group satisfying the 8-condition. Then $Sm(G) \subseteq PO(G)$.*

Proof. Take an element $U - V \in Sm(G)$. We may assume that the G-modules U and V are Smith equivalent, i.e., there exists a smooth action of G on S^n with $(S^n)^G = \{x, y\}$ where $T_x(S^n) \cong U$ and $T_y(S^n) \cong V$.

By [AB], [Mi], [Sa], U and V are isomorphic when restricted to any p-subgroup of G for any odd prime p dividing the order of G, and by elementary character theory arguments, U and V are isomorphic when restricted to any cyclic subgroup of G of order 2 or 4.

Assume G has no element of order 8. Then it follows that U and V are isomorphic when restricted to any cyclic 2-subgroup (and thus any 2-subgroup) of G.

Assume G has an element of order 8 and for any cyclic 2-subgroup H of G of order ≥ 8, $\dim W^H > 0$ for all irreducible G-modules W. By the Slice Theorem and Smith theory, $\dim(S^n)^H = \dim U^H = \dim V^H > 0$ and $(S^n)^H$ is connected, so U and V are isomorphic as H-modules. Thus U and V are isomorphic when restricted to any 2-subgroup of G.

So, we have noted that $U - V \in IO(G)$, and thus $U - V \in PO(G)$ by the Slice Theorem, proving that $Sm(G) \subseteq PO(G)$. □

Corollary 6.2. *Let G be a finite group satisfying the 8-condition, and assume that $r_G \leq 1$. Then $Sm(G) = 0$.*

Proof. As G satisfies the 8-condition, $Sm(G) \subseteq PO(G)$ by Lemma 6.1. By assumption, $r_G \leq 1$ and thus $PO(G) = 0$ by Lemma 0.1. Therefore $Sm(G) = 0$. □

Corollary 6.3. *Let G be a finite perfect group. Then $PO(G) \subseteq Sm(G)$. If G also satisfies the 8-condition, then $Sm(G) = PO(G)$.*

Proof. As G is perfect, $LO(G) = PO(G)$ and thus $PO(G) \subseteq Sm(G)$ by Theorem 5.1. So, if G also satisfies the 8-condition, then $Sm(G) = PO(G)$ by Lemma 6.1. □

7. Proofs of Theorems A1–A3, B1–B3, C1–C3, and Concluding Corollary

Proof of Theorem A1. Let G be a finite group and let H be a normal subgroup of G such that $G/H \cong \mathbb{Z}_{pq}$ for two distinct odd primes p and q. We shall prove that $r_G \geq 2$ and $LO(G) \neq 0$. First, we note that $\mathbb{Z}_{pq}$ contains $\phi(pq) = (p-1)(q-1)$ elements of order pq, and hence $\frac{1}{2}(p-1)(q-1)$ real conjugacy classes of elements of order pq. Since we may assume that $p \geq 3$ and $q \geq 5$, and we know that $r_G \geq \overline{r}_{G/H} \geq r_{G/H}$, thus $r_G \geq 4$.

Now, we prove that $LO(G) \neq 0$. Set $n = pq$. Let ζ_n be the primitive n-th root of unity. Assume that $H = 1$ so that $G = \mathbb{Z}_n = \langle\, g \,|\, g^n = 1 \,\rangle$. Take $U = U_1 \oplus U_2$ and $V = V_1 \oplus V_2$, where U_i and V_i $(i = 1, 2)$ are the irreducible 1-dimensional complex G-modules with characters

$$\chi_U(g) = \chi_{U_1}(g) + \chi_{U_2}(g) = \zeta_n + \zeta_n^2$$
$$\chi_V(g) = \chi_{V_1}(g) + \chi_{V_2}(g) = \zeta_n^a + \zeta_n^b,$$

and the integers a and b are chosen so that the following holds:

$$a \equiv 1 \pmod p, \quad a \equiv 2 \pmod q$$
$$b \equiv 2 \pmod p, \quad b \equiv 1 \pmod q$$

(for example, if $p = 3$ and $q = 5$, $a = 7$ and $b = 11$). Then U and V are complex $\mathcal{L}$-free G-modules isomorphic when restricted to $P \cong \mathbb{Z}_p$ or $\mathbb{Z}_q$. The realifications $r(U)$ and $r(V)$ are not isomorphic as real G-modules (remember p and q are odd) but $r(U)$ and $r(V)$ are isomorphic when restricted to $P \cong \mathbb{Z}_p$ or $\mathbb{Z}_q$. Thus, $r(U) - r(V)$ is a nonzero element of $LO(G)$. Now, if $H \neq 1$, the epimorphism $G \to G/H \cong \mathbb{Z}_n$ (mapping large subgroups of G onto large subgroups of G/H) allows us to consider the complex $\mathbb{Z}_n$-modules U and V constructed above as complex $\mathcal{L}$-free G-modules, and as before, $r(U) - r(V)$ is a nonzero element of $LO(G)$. □

Proof of Theorem A2. Let G be a finite Oliver group of odd order. We shall prove that $r_G \geq 2$ and $LO(G) \neq 0$. In the case G has a cyclic quotient of order pq for two distinct odd primes p and q, the result follows by Theorem A1. In turn, in the case each cyclic quotient of G has prime power order, $r_G > \overline{r}_{G/G^{\mathrm{nil}}}$ and $r_G \geq 2$ by Proposition 1.6, and thus $IO(G, G^{\mathrm{nil}}) \neq 0$ by Lemma 0.1, and hence $LO(G) \neq 0$ by Lemma 0.3. □

Proof of Theorem A3. Let G be a finite nonsolvable group. We shall prove that $LO(G) = 0$ for $r_G \leq 1$, and $LO(G) \neq 0$ for $r_G \geq 2$, except when $G \cong \mathrm{Aut}(A_6)$ or $P\Sigma L(2, 27)$. For $r_G \leq 1$, $PO(G) = 0$ by Lemma 0.1, and thus $LO(G) = 0$ by Lemma 0.3, while for $r_G \geq 2$, $IO(G, G^{sol}) \neq 0$ by Corollary 3.12, and thus $LO(G) \neq 0$ by Lemma 0.3, except when $G \cong \mathrm{Aut}(A_6)$ or $P\Sigma L(2, 27)$. In the case $G = \mathrm{Aut}(A_6)$, the Dress subgroup $G^p = G$ for any prime $p \neq 2$, and $G^2 = A_6 = G^{\mathrm{sol}}$. Hence $IO(G, G^2) = 0$ by Corollary 3.12, and thus $LO(G) = 0$ by Lemma 0.3. In turn, if $G = P\Sigma L(2, 27)$, the Dress subgroup $G^p = G$ for any prime $p \neq 3$, and $G^3 = PSL(2, 27) = G^{\mathrm{sol}}$. Hence $IO(G, G^3) = 0$ by Corollary 3.12, and thus $LO(G) = 0$ by Lemma 0.3. Finally, by inspection in [C–W], we see that for $G = \mathrm{Aut}(A_6)$, $r_G = 2$ corresponding to elements of order 6 and 10, and we see that for $G = P\Sigma L(2, 27)$, $r_G = 2$ corresponding to elements of order 6 and 14 . □

Proof of Theorem B1. Let G be a finite Oliver group with a cyclic quotient of order pq for two distinct odd primes p and q. Then $G^p \neq G$ and $G^q \neq G$, and thus G is a gap group by [MSY]. Moreover, $r_G \geq 2$ and $LO(G) \neq 0$ by Theorem A1. As $LO(G) \subseteq Sm(G)$ by Theorem 5.1, we see that $Pm(G) \neq 0$, proving Theorem B1. □

Proof of Theorem B2. Let G be a finite Oliver group of odd order. Then $G^2 = G$, and thus G is a gap group by [MSY]. Moreover, $r_G \geq 2$ and $LO(G) \neq 0$ by Theorem A2. As $LO(G) \subseteq Sm(G)$ by Theorem 5.1, we see that $Pm(G) \neq 0$, proving Theorem B2. □

Proof of Theorem B3. Let G be a finite nonsolvable gap group. By [MSY], $\mathrm{Aut}(A_6)$ is not a gap group. So, G is not isomorphic to $\mathrm{Aut}(A_6)$. Assume also that G is not isomorphic to $P\Sigma L(2,27)$. Then, for $r_G \leq 1$, $PO(G) = 0$ by Lemma 0.1, and therefore $Pm(G) = 0$. In turn, for $r_G \geq 2$, $LO(G) \neq 0$ by Theorem A3, and thus it follows from Theorem 5.1 that $Pm(G) \neq 0$. Hence, $Pm(G) \neq 0$ if and only if $r_G \geq 2$, proving Theorem B3. □

Proof of Theorem C1. For a finite nonabelian simple group G, we shall prove that the claims (1) and (2) of Theorem C1 both hold. For $r_G \leq 1$, Corollary 2.20 asserts that G is isomorphic to one the groups listed in (1). Some inspection in [C–W] or [GLS] confirms that $r_G = 0$ for $G \cong PSL(2,q)$ with $q = 5,7,8,9,17$, and for $G \cong PSL(3,4)$, $Sz(8)$, or $Sz(32)$. Moreover, $r_G = 1$ corresponding to an element of order 6 when $G \cong PSL(2,11)$, $PSL(2,13)$, $PSL(3,3)$, A_7, M_{11}, or M_{22}. In addition, G has an element of order 8 if and only if $G \cong PSL(2,17)$, $PSL(3,3)$, M_{11}, or M_{22}. Further inspection in [C–W] or [GLS] shows that for $r_G \leq 1$, G satisfies the 8-condition. Hence, $Sm(G) = 0$ by Corollary 6.2, proving (1). In general, G is simple and nonabelian, and so G is a gap group by [MSY]. Hence, for $r_G \geq 2$, $Pm(G) \neq 0$ by Theorem B3, and thus $Sm(G) \neq 0$, proving (2). □

Proof of Theorem C2. Let $G = SL(n,q)$ or $Sp(n,q)$ where $n \geq 2$ and n is even in the latter case, and q is any prime power in both cases. We shall prove that the claims (1) and (2) of Theorem C2 both hold. Clearly, $r_G = 0$ when $G = SL(2,2) \cong PSL(2,2) \cong S_3$, and by Theorem C1, $r_G = 0$ when $G = SL(2,4) \cong PSL(2,4) \cong PSL(2,5) \cong A_5$, as well as when $G = SL(2,8) \cong PSL(2,8)$ or $G = SL(3,2) \cong PSL(3,2) \cong PSL(2,7)$. Moreover, for $G = SL(3,3) \cong PSL(3,3)$, $r_G = 1$ corresponding to an element of order 6. The same is true for $G = SL(2,3)$ because G has elements of orders 1, 2, 3, 4, and 6, and the elements of order 6 are all real conjugate in G (cf. [LP, Proposition 2.3]).

Note that neither $SL(2,2)$ nor $SL(2,3)$ is an Oliver group and recall that except for $SL(2,2)$ and $SL(2,3)$, every $SL(n,q)$ is a perfect (and thus Oliver) group. So, by

Theorem 2.1, $r_G \geq 2$ when $G = SL(n, q)$, except when $G = SL(2, 2)$, $SL(2, 4)$, $SL(2, 8)$, $SL(2, 3)$, $SL(3, 2)$, or $SL(3, 3)$ (where $r_G \leq 1$ as we noted above). In the exceptional cases, G has an element of order 8 if and only if $G = SL(3, 3)$, and by checking in [C–W] or [GLS], we see that $SL(3, 3) \cong PSL(3, 3)$ satisfies the 8-condition.

As $Sp(2, q) \cong SL(2, q)$ for any prime power q, we see that $r_G \leq 1$ when $G = Sp(2, 2)$, $Sp(2, 4)$, $Sp(2, 8)$, or $Sp(2, 3)$. Except for $Sp(2, 2)$, $Sp(2, 3)$, and $Sp(4, 2)$, every $Sp(n, q)$ is a perfect (and thus Oliver) group. Moreover, $Sp(4, 2) \cong S_6$ is a nonsolvable (and thus Oliver) group. So, by Theorem 2.1, $r_G \geq 2$ when $G = Sp(n, q)$, except when $G = Sp(2, 2)$, $Sp(2, 4)$, $Sp(2, 8)$, or $Sp(2, 3)$ (where $r_G \leq 1$ as we noted above).

In effect, for $G = SL(n, q)$ or $Sp(n, q)$ with $r_G \leq 1$, G satisfies the 8-condition. Hence, $Sm(G) = 0$ by Corollary 6.2, proving (1). To prove (2), note that for $r_G \geq 2$, G is either a perfect (and thus gap) group (cf. [MSY]) or $G = Sp(4, 2) \cong S_6$ and S_6 is a gap group by [DH] or [MSY]. Therefore, $Pm(G) \neq 0$ by Theorem B3, and so $Sm(G) \neq 0$. □

Proof of Theorem C3. For $G = A_n$ or S_n with $n \geq 2$, we shall prove that the claims (1) and (2) of Theorem C3 both hold. First, we consider the case $G = A_n$. For $n \leq 6$, $r_G = 0$ because each element of G has prime power order. For $n = 7$, $r_G = 1$ corresponding to the element (12)(34)(567) of order 6. For $n \geq 8$, $r_G \geq 2$ because the elements (12)(34)(567) and (123456)(78) have order 6 and are not real conjugate in G.

Now, we consider the case $G = S_n$. For $n \leq 4$, $r_G = 0$ because each element of G has prime power order. For $n = 5$, $r_G = 1$ corresponding to the element (12)(345) of order 6. For $n \geq 6$, $r_G \geq 2$ because the elements (12)(345) and (123456) have order 6 and are not real conjugate in G. As a result, for $G = A_n$ or S_n with $r_G \leq 1$, G is one of the groups listed in (1). For $G = A_n$ or S_n with $n \leq 7$, G has no element of order 8 because any permutation of order 8 must involve an 8-cycle in its cycle decomposition.

In effect, for $r_G \leq 1$, G satisfies the 8-condition. Hence, $Sm(G) = 0$ by Corollary 6.2, proving (1). For $G = A_n$ with $n \geq 5$, G is simple and nonabelian, and so G is a gap group by [MSY]. For $G = S_n$ with $n \geq 6$, G is a gap group by [DS] or [MSY]. In particular, for $r_G \geq 2$, G is a gap group and thus $Pm(G) \neq 0$ by Theorem B3, proving (2). □

Proof of Concluding Corollary. We shall prove that for G as in the Concluding Corollary, $Sm(G) = 0$ if and only if $r_G \leq 1$. First, note that the assertion holds for any finite simple group G. In fact, if G is nonabelian, the assertion follows from Theorem C1, and if $G \cong \mathbb{Z}_p$ for some prime p, $r_G = 0$ and $Sm(G) = 0$ by [AB], [Mi]. Now, note that for $G = SL(n, q)$, $Sp(n, q)$, A_n, or S_n, the assertion follows from Theorems C2 and C3.

The assertion holds also for $G = PSL(n, q)$. In fact, except for $PSL(2, 2)$ and $PSL(2, 3)$, every $PSL(n, q)$ is a simple group, and thus Theorem C1 proves the assertion. Moreover, for $G = PSL(2, 2) \cong S_3$ or $G = PSL(2, 3) \cong A_4$, G has no element of order 8 and each element of G has a prime order, and so $r_G = 0$. Hence, $Sm(G) = 0$ by Corollary 6.2.

Finally, we note that for $G = PSp(n, q)$, $Sm(G) = 0$ if and only if $r_G \leq 1$. In fact, except for $PSp(2, 2)$, $PSp(2, 3)$, and $PSp(4, 2)$, every $PSp(n, q)$ is a simple group, and thus with the exceptions noted, the assertion follows from Theorem C1. In the exceptional cases, $PSp(2, 2) \cong PSL(2, 2) \cong S_3$, $PSp(2, 3) \cong PSL(2, 3) \cong A_4$, and $PSp(4, 2) \cong Sp(4, 2) \cong S_6$, and thus the assertion follows by the arguments above. □

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