Smooth actions of finite Oliver groups on spheres

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Abstract

In this article, we deal with the following two questions. For smooth actions of a given finite group G on spheres S, which smooth manifolds F occur as the fixed point sets in S, and which real G-vector bundles ν over F occur as the equivariant normal bundles of F in S? We focus on the case G is an Oliver group and answer both questions under some conditions imposed on G, F, and ν. We construct smooth actions of G on spheres by making use of equivariant surgery, equivariant thickening, and Oliver's equivariant bundle extension method modified by an equivariant wegde sum construction and an equivariant bundle subtraction procedure.

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Introduction

In the transformation group theory, a basic problem reads as follows. For smooth actions of a given finite (more generally, compact Lie) group G on

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specific smooth manifolds M, describe necessary and sufficient conditions for a smooth manifold F to occur as (i.e., to be diffeomorphic to) the fixed point set M^G. Once we know that $F = M^G$, another basic problem is to describe necessary and sufficient conditions for a real G-vector bundle ν over F to occur as (i.e., to be isomorphic to) the equivariant normal bundle $\nu_{F \subset M}$ of F in M.

By the Slice Theorem, for a compact Lie group G and a smooth G-manifold M, the fixed point set $F = M^G$ is a smooth submanifold of M with boundary $\partial F = F \cap \partial M$ (cf. [6, Corollary 2.5, p. 309]). If M is compact, F is compact. Thus, if M is closed (i.e., M is compact and $\partial M = \varnothing$), F is closed too.

In this article, by a disk D (resp., sphere S) we mean a ball with boundary (resp., the boundary of a ball) in some Euclidean space, considered with the standard smooth structure. We deal with the basic two problems for smooth actions of G on spheres S. In particular, the fixed point set $F = S^G$ is a closed smooth manifold. Further restrictions on F may occur and they depend on the group G. We focus on the case G is a finite Oliver group.

Let G be a finite group. Following [41], a series of subgroups of G of the form $P \trianglelefteq H \trianglelefteq G$ is called an *isthmus series* if $|P| = p^m$ and $|G/H| = q^n$ for some primes p and q (possibly $p = q$) and some integers $m, n \geq 0$, and the quotient group H/P is cyclic (possibly $H = P$).

According to Oliver [33], a finite group G has a smooth fixed point free action on a disk if and only if G has no isthmus series of subgroups. More generally, by the results of Oliver [33] and [34], a compact Lie group G has a smooth fixed point free action on a disk if and only if the identity connected component G_0 of G is nonabelian or G/G_0 has no isthmus series of subgroups.

A finite group G is called an *Oliver group* if G has no isthmus series of subgroups (i.e., G has a smooth fixed point free action on a disk). Recall that a finite nilpotent group G is an Oliver group if and only if G has three or more noncyclic Sylow subgroups, and any finite nonsolvable group G is an Oliver group (cf. [33] and [34]). Recall also that any finite nonsolvable group G has a smooth action on a sphere with exactly one fixed point (cf. [24]). Moreover, a finite group G has a smooth action on a sphere with exactly one fixed point if and only if G is an Oliver group (cf. [25]). As a result, for a finite group G, the following three claims are equivalent.

(1) G has a smooth one fixed point action on a sphere.
(2) G has a smooth fixed point free action on a disk.
(3) G is an Oliver group.

For a given finite group G, we denote by $\mathcal{P}(G)$ the family of subgroups of G consisting of the trivial subgroup $\{e\}$ and all p-subgroups for all primes $p \,\big|\, |G|$.

A subgroup H of a finite group G is called a *large subgroup* of G if $O^p(G) \leq H$ for some prime p, where $O^p(G)$ is the smallest normal subgroup of G such that $|G/O^p(G)| = p^k$ for some integer $k \geq 0$. We denote by $\mathcal{L}(G)$ the family of large subgroups of G.

By a real G-*module* we mean a finite dimensional real vector space V with a linear action of G. For a given family $\mathcal{L}$ of subgroups of G, a real G-module V is called $\mathcal{L}$-*free* if $\dim V^H = 0$ for each $H \in \mathcal{L}$. More generally, a real G-vector bundle ν over a smooth G-manifold M is called $\mathcal{L}$-*free* if no $H \in \mathcal{L}$ occurs as the isotropy subgroup in the total space $S(\nu)$ of the invariant unit sphere bundle of ν.

Except for Theorems 11–13 involving $\mathcal{L}$-free G-vector bundles for $\mathcal{L} = \{G\}$, we consider (without mentioning it explicitly) $\mathcal{L}$-free G-vector bundles *always* for $\mathcal{L} = \mathcal{L}(G)$. Note that for any finite perfect group G, $O^p(G) = G$ for each prime p, and thus $\mathcal{L}(G) = \{G\}$.

Following [32], a finite group G is called a *gap group* if $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ and G has a real $\mathcal{L}$-free G-module V such that $\dim V^P > 2 \dim V^H$ for all subgroups $P < H \leq G$ with $P \in \mathcal{P}(G)$. If G is a finite Oliver group, then $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ (cf. [25]). We refer the reader to [14], [32], and [49] for basic information on gap groups, in particular, for arguments showing that a finite Oliver group G is a gap group under either of the following conditions.

(1) G has a cyclic quotient of order pq for two distinct odd primes p and q (which is true when G is nilpotent, in particular, when G is abelian).
(2) $O^2(G) = G$ (which is true when G is perfect or G is of odd order).
(3) G has a quotient which is a gap group.

Also, by [14] or [32], the symmetric group S_n on n letters is a gap group if and only if $n \geq 6$. Let H be a subgroup of a finite group G. If H is an Oliver group, then G itself is an Oliver group. However, if H is a gap group, then G is not necessarily a gap group (e.g., take $H = A_5$ and $G = S_5$).

Let G be a finite group not of prime power order. An action of G on a space X is called $\mathcal{P}$-*typical* if $X^P \smallsetminus X^G \neq \varnothing$ for each $P \in \mathcal{P}(G)$, which amounts to saying that $X^{G_p} \smallsetminus X^G \neq \varnothing$ for each Sylow subgroup G_p of G with $p \,\big|\, |G|$.

Let G be a finite group not of prime power order (resp., a finite Oliver group). By Smith theory, any smooth fixed point free action of G on a Euclidean space (resp., disk) is $\mathcal{P}$-typical. Moreover, for a finite Oliver group G, any smooth one fixed point action of G on a sphere is $\mathcal{P}$-typical.

Any smooth action of G on a Euclidean space (resp., disk) M with nonempty fixed point set F can be converted into a smooth $\mathcal{P}$-typical action of G on some Euclidean space (resp., disk) with the same fixed point set F. In fact, the diagonal action of G on $M \times V$ (resp., $M \times D(V)$) is the required action of G, where V (resp., $D(V)$) is a real G-module (resp., the invariant unit disk in a real G-module V) with $\dim V^G = 0$ and $\dim V^P > 0$ for each $P \in \mathcal{P}(G)$.

In the next section, for any finite group G not of prime power order, and any real G-vector bundle η over a smooth manifold F, we define a KO-theory obstruction $\mathcal{O}\ell(\eta)$ which we shall call the *Oliver obstruction* of η. In the case $\eta = \tau_F \oplus \nu$, the Whitney sum of the tangent bundle τ_F of F and a real G-vector bundle ν over F, we set $\mathcal{O}\ell(F, \nu) = \mathcal{O}\ell(\tau_F \oplus \nu)$.

Now, we are ready to state Theorems 1–6 which contain the main results of this article. Theorems 1 and 2 correspond to Theorems 11 and 12, where using the notion of Oliver obstruction, we restate Oliver's results about the fixed point sets of smooth actions on disks and Euclidean spaces.

Theorem 1 *Let G be a finite Oliver gap group. Let F be a smooth manifold whose connected components are all simply connected. Let ν be a real $\mathcal{L}$-free G-vector bundle over F. Then the following two claims are equivalent.*

(1) F is the fixed point set of a smooth $\mathcal{P}$-typical action of G on a sphere S such that $\nu_{F\subset S} \cong \nu \oplus \varepsilon_F^W$ for some real $\mathcal{L}$-free G-module W.
(2) F is closed and $\mathcal{O}\ell(F, \nu) = 0$.

Theorem 2 *Let G be a finite Oliver gap group. Let F be a smooth manifold whose connected components are all simply connected. Assume also that F is closed. Then the following two claims are equivalent.*

(1) F is the fixed point set of a smooth $\mathcal{P}$-typical action of G on a sphere S such that the equivariant normal bundle $\nu_{F\subset S}$ is $\mathcal{L}$-free.
(2) There exists a real $\mathcal{L}$-free G-vector bundle ν over F with $\mathcal{O}\ell(F, \nu) = 0$.

Let G be a finite group. As in [36, p. 599], two real G-modules U and V are called *$\mathcal{P}$-matched* if $\mathrm{Res}^G_P(U) \cong \mathrm{Res}^G_P(V)$ for each $P \in \mathcal{P}(G)$. The following theorem corresponds to Theorem 13 concerned with smooth actions on disks and Euclidean spaces.

Theorem 3 *Let G be a finite Oliver gap group. Let F be a closed smooth manifold whose connected components $F_1, \ldots, F_k$ are all stably parallelizable. Let $V_1, \ldots, V_k$ be real G-modules with $\dim V_i^H = \dim F_i$ for each $H \in \mathcal{L}(G)$ and $1 \le i \le k$. Then the following two claims are equivalent.*

(1) F is the fixed point set of a smooth $\mathcal{P}$-typical action of G on a sphere S such that $T_{x_i}(S) \cong V_i \oplus W$ for each $x_i \in F_i$ and $1 \le i \le k$, and some real $\mathcal{L}$-free G-module W.
(2) The G-modules V_i and V_j are $\mathcal{P}$-matched for $1 \le i, j \le k$.

Now, for a finite Oliver group G and a closed smooth manifold F, we wish to consider the following three claims about G and F.

(1) F is the fixed point set of a smooth $\mathcal{P}$-typical action of G on a sphere.
(2) F is the fixed point set of a smooth action of G on a disk.
(3) F is the fixed point set of a smooth action of G on a Euclidean space.

By Theorem 11, (2) and (3) are equivalent (cf. the comment after Theorem 11). By Lemma 7 and Theorem 11, (1) implies (2). In general, the question whether (2) implies (1) is open. However, under some additional conditions imposed on G and F, we show that (2) implies (1), and thus (1) and (2) are equivalent.

In the case the claims (1) and (2) are equivalent, we can answer explicitly (by using Theorem 12) the question which closed smooth manifolds F occur as the fixed point sets of smooth $\mathcal{P}$-typical actions of G on spheres.

Theorem 4 *Let G be a finite (nontrivial) perfect group. Let F be a closed smooth manifold such that each connected component of F is simply connected or stably parallelizable. Then the following three claims are equivalent.*

(1) F is the fixed point set of a smooth $\mathcal{P}$-typical action of G on a sphere.
(2) F is the fixed point set of a smooth action of G on a disk.
(3) The conclusion about the class $[\tau_F]$ in Theorem 12 (3) holds.

We say that a finite group G has a *pqr-cyclic quotient* if G has a cyclic quotient of order pqr for three distinct primes p, q, and r. Recall that a finite group G is nilpotent if and only if G is the product of its Sylow subgroups (i.e., all Sylow subgroups of G are normal subgroups of G). Thus, if a finite nilpotent group G has three or more Sylow subgroups, then G has a pqr-cyclic quotient. Therefore, any finite nilpotent Oliver group G has a pqr-cyclic quotient.

Theorem 5 *Let G be a finite Oliver group with a pqr-cyclic quotient, and in the case G is of even order, assume also that $G_2 \trianglelefteq G$. Let F be a closed smooth manifold such that each connected component of F is simply connected or stably parallelizable. Then the following three claims are equivalent.*

(1) F is the fixed point set of a smooth $\mathcal{P}$-typical action of G on a sphere.
(2) F is the fixed point set of a smooth action of G on a disk.
(3) F is a stably complex manifold.

The question which configurations of Pontrjagin numbers occur for the fixed point sets of smooth actions of G on spheres was considered for the first time by Schultz [47] in the case $G = \mathbb{Z}_{pq}$, the cyclic group of order pq for two relatively prime odd integers p and q. By [40, Theorem A], all configurations of Chern and Pontrjagin numbers do occur when G is a finite perfect group with appropriate cyclic subgroups. In this article, we show that the same is true when G is a finite perfect group with a pq-element, or G is a finite Oliver group with a pqr-cyclic quotient. This extends the result of [40, Theorem A].

Theorem 6 *Let G be a finite perfect group with a pq-element, or let G be a finite Oliver group with a pqr-cyclic quotient. Let M be an oriented closed smooth manifold of dimension $2k$ (resp., $4k$) for an integer $k \geq 0$. Then there exists a smooth action of G on some sphere such that the fixed point set F is an oriented closed smooth manifold of dimension $2k$ (resp., $4k$) and F has the same Chern (resp., Pontrjagin) numbers as does M.*

The proofs of Theorems 1–6 follow by a number of results collected in this article. In the proofs, the crucial new ingredients are constructions of group actions described in Theorems 27 and 28, and Corollary 29. In order to perform the constructions, we use equivariant thickening (Theorem 17), equivariant surgery (Theorem 18), an equivariant bundle extension method (Theorem 19) and an equivariant bundle subtraction procedure (Theorem 20).

We refer the reader to the textbooks [1], [6], [12], [21] for background material on the transformation group theory that we use in this article.

1 Oliver obstruction and stably complex fixed point sets

Let G be a finite group not of prime power order. Let F be a smooth manifold with the trivial action of G. We introduce the notion of Oliver obstruction for a real G-vector bundle η over F. In order to define the obstruction, for any $P \in \mathcal{P}(G)$, we consider the P-vector bundle $\mathrm{Res}^G_P(\eta)$ obtained from η by restricting the action of G to the action of P.

Let $\widetilde{KO}(F)$ be the reduced real K-theory of F. Let $[\mathrm{Res}^G_{\{e\}}(\eta)] \in \widetilde{KO}(F)$ be the class of $\mathrm{Res}^G_{\{e\}}(\eta)$. For any p-subgroup $P \neq \{e\}$, let $\widetilde{KO}_P(F)_{(p)}$ be the reduced P-equivariant real K-theory of F localized at p, and let $\widetilde{KO}_P(F)_{(p)}/\mathrm{div}_p$ be the quotient of $\widetilde{KO}_P(F)_{(p)}$ by the subgroup of infinitely p-divisible elements. Let $[\mathrm{Res}^G_P(\eta)]_{(p)} \in \widetilde{KO}_P(F)_{(p)}/\mathrm{div}_p$ be the element determined by $\mathrm{Res}^G_P(\eta)$.

Define the *Oliver obstruction* $\mathcal{O}\ell(\eta)$ of η by setting

$$\mathcal{O}\ell(\eta) = [\mathrm{Res}^G_{\{e\}}(\eta)] + \sum_{P \neq \{e\}} [\mathrm{Res}^G_P(\eta)]_{(p)} \;\in\; \widetilde{KO}(F) \oplus \bigoplus_{P \neq \{e\}} \widetilde{KO}_P(F)_{(p)}/\mathrm{div}_p$$

where $P \neq \{e\}$ ranges over all p-subgroups of G for all primes $p \,\big|\, |G|$.

If F is compact, $\widetilde{KO}_P(F)$ is finitely generated, and therefore the subgroup of infinitely p-divisible elements in $\widetilde{KO}_P(F)_{(p)}$ is trivial. As a result,

$$\widetilde{KO}_P(F)_{(p)}/\mathrm{div}_p = \widetilde{KO}_P(F)_{(p)}$$

and thus $\mathcal{O}\ell(\eta) = 0$ if and only if $\mathrm{Res}^G_{\{e\}}(\eta)$ stably is a product vector bundle and the element $[\mathrm{Res}^G_P(\eta)] \in \widetilde{KO}_P(F)$ is of finite order not divisible by p for each prime $p \,\big|\, |G|$ and each p-subgroup $P \neq \{e\}$ of G.

For a real G-vector bundle ν over F, we consider the Whitney sum $\tau_F \oplus \nu$ of τ_F and ν, where τ_F is the tangent bundle of F with the trivial action of G, and we set $\mathcal{O}\ell(F, \nu) = \mathcal{O}\ell(\tau_F \oplus \nu)$.

According to [36], if a finite group G acts smoothly on a disk or Euclidean space M, then $\mathcal{O}\ell(F, \nu) = 0$ for $F = M^G$ and $\nu = \nu_{F \subset M}$. We show that a similar result holds for smooth $\mathcal{P}$-typical actions of G on homotopy spheres. Henceforth, by a *homotopy sphere* we mean a topological sphere with a smooth structure. Recall that each homotopy sphere is a stably parallelizable manifold (called also a π-manifold; see [22] and [23, Corollary (8.6), p. 191]).

Lemma 7 *Let G be a finite group not of prime power order. Let F be the fixed point set of a smooth $\mathcal{P}$-typical action of G on a homotopy sphere Σ. Then $\mathcal{O}\ell(F, \nu) = 0$ for $\nu = \nu_{F \subset \Sigma}$.*

PROOF. As Σ is stably parallelizable, $[\mathrm{Res}^G_{\{e\}}(\tau_F \oplus \nu)] = [\mathrm{Res}^G_{\{e\}}(\tau_\Sigma|_F)] = 0$ in $\widetilde{KO}(F)$. For any prime $p \,\big|\, |G|$, take any p-subgroup $P \neq \{e\}$ of G and choose some point $x \in \Sigma^P \smallsetminus F$ (remember the action of G on Σ is $\mathcal{P}$-typical). Now, using the Slice Theorem, remove some sufficiently small open P-invariant ball neighborhood U of x in Σ to get a smooth action of P on the disk $D = \Sigma \smallsetminus U$ with $D^P \supset F$. It follows from Smith theory that $\widetilde{KO}_P(D^P)_{(p)} = 0$. Therefore $[\mathrm{Res}^G_P(\tau_F \oplus \nu)] = [\mathrm{Res}^G_P((\tau_M|_{D^P})|_F)] = 0$ in $\widetilde{KO}_P(F)_{(p)}$. $\square$

Following [18], a smooth manifold M is called *stably complex* if the class $[\tau_M]$ determined by τ_M in $\widetilde{KO}(M)$ lies in the image of the forgetful (realification) map $r_{\mathbb{C}} : \widetilde{K}(M) \to \widetilde{KO}(M)$ defined on the reduced complex K-theory $\widetilde{K}(M)$ of M; i.e., the tangent bundle τ_M of M admits a complex structure possibly after adding a product vector bundle. This condition amounts to saying that M has a smooth embedding into some Euclidean space, such that the normal bundle of the embedding admits a complex structure. In particular, a stably complex smooth manifold M is orientable and the connected components of M are all either odd or even dimensional.

The following proposition goes back to Edmonds and Lee [18, (3.1) and (3.2)] and it involves a Sylow 2-subgroup G_2 of G.

Proposition 8 *If a finite group G acts smoothly on a smooth manifold M, then the fixed point set M^G is a stably complex manifold under either of the following two conditions.*

(1) G is of odd order and M is stably complex.
(2) $G_2 \trianglelefteq G$ and M is $\mathbb{Z}_2$-acyclic (and thus stably complex).

Now, we wish to show that if a finite group G acts smoothly on a homotopy sphere Σ, then Σ^G is a stably complex manifold under the condition that $G_2 \trianglelefteq G$ and $\Sigma^{G_2} \neq \Sigma^G$. First, we prove the following lemma.

Lemma 9 *If a finite 2-group G acts smoothly on a homotopy sphere Σ, then for any point $x \in \Sigma^G$, the manifold $\Sigma^G \smallsetminus \{x\}$ is stably complex.*

PROOF. For any $x \in \Sigma^G$, consider the action of G on the contractible manifold $E = \Sigma \smallsetminus \{x\}$. As G is a finite 2-group, it follows from Smith theory that E^G is $\mathbb{Z}_2$-acyclic, and thus E^G is stably complex (cf. [18], [20]). Clearly, $E^G = \Sigma^G \smallsetminus \{x\}$. □

Proposition 10 *If a finite group G acts smoothly on a homotopy sphere Σ, then the fixed point set Σ^G is a stably complex manifold under either of the following two conditions.*

(1) G is of odd order.
(2) $G_2 \trianglelefteq G$ and $\Sigma^{G_2} \neq \Sigma^G$.

PROOF. (1) Assume that G is of odd order. As Σ is stably parallelizable, Σ is stably complex, and thus so is Σ^G by Proposition 8 (1).

(2) Assume that $G_2 \trianglelefteq G$ and $\Sigma^{G_2} \neq \Sigma^G$. Choose some point $x \in \Sigma^{G_2} \smallsetminus \Sigma^G$. As $G_2 \trianglelefteq G$, the orbit $G(x)$ of x is a subset of Σ^{G_2}. Consider the standard action of G/G_2 on Σ^{G_2} with $(\Sigma^{G_2})^{G/G_2} = \Sigma^G$. By Lemma 9, the manifold $M = \Sigma^{G_2} \smallsetminus G(x)$ is stably complex. As the group G/G_2 (of odd order) acts smoothly on M with $M^{G/G_2} = \Sigma^G$, the manifold Σ^G is stably complex by Proposition 8 (1). □

2 Fixed point sets and orientability of equivariant bundles

Let G be a p-group for some prime p. Then a compact smooth manifold F is the fixed point set of a smooth action of G on a disk if and only if F is $\mathbb{Z}_p$-acyclic and stably complex. The necessity of the conditions on F follows from Smith theory and Proposition 8, while the sufficiency reduces to the case

$G = \mathbb{Z}_p$ and goes back to Jones [20]. Also, a smooth manifold F with $\partial F = \varnothing$ is the fixed point set of a smooth action of G on a Euclidean space if and only if F is $\mathbb{Z}_p$-acyclic and stably complex (cf. [39, Theorem A]). We refer the reader to the article of Weinberger [50] for a survey of related results for actions of p-groups on spheres.

In the case G is a finite group not of prime power order, Oliver [36] has answered completely the following two basic questions. For smooth actions of G on disks (resp., Euclidean spaces) M, which smooth manifolds F occur as (i.e., are diffeomorphic to) the fixed point sets M^G (written $F = M^G$) and which real G-vector bundles ν over F stably occur as the equivariant normal bundles of F in M (i.e., as G-vector bundles, $\nu_{F \subset M} \cong \nu \oplus \varepsilon_F^W$ for some real G-module W)? We refer the reader to the articles [2], [3], [13], [17]–[19], [33]–[35], [37]–[40], [46] for partial answers and related results obtained before the work of Oliver [36]. In Theorems 11 and 12, using the notion of Oliver obstruction, we restate the main results of [36] about the fixed point sets of smooth actions of G on disks and Euclidean spaces (cf. Theorems 1 and 2 in this article).

Theorem 11 ([36]) *Let G be a finite group not of prime power order, and let $\mathcal{L} = \{G\}$. Let F be a smooth manifold and let ν be a real $\mathcal{L}$-free G-vector bundle over F. Then the following two claims are equivalent.*

(1) F is the fixed point set of a smooth action of G on a disk (resp., Euclidean space) M such that $\nu_{F \subset M} \cong \nu \oplus \varepsilon_F^W$ for some real $\mathcal{L}$-free G-module W.

(2) F is compact, the Euler characteristic $\chi(F) \equiv 1 \pmod{n_G}$ and the Oliver obstruction $\mathcal{O}\ell(F, \nu) = 0$ (resp., $\partial F = \varnothing$ and $\mathcal{O}\ell(F, \nu) = 0$).

The integer $n_G \geq 0$ (known as the *Oliver number* of G) has been determined and computed by Oliver [33]–[35] for all finite groups G not of prime power order (see [36, Theorem 0.3] for a summary of computation of n_G). By [33], $n_G = 1$ if and only if G has a smooth fixed point free action on a disk. Hence, $n_G = 1$ if and only if G is an Oliver group. So, if G is a finite Oliver group, then there is no restriction on the Euler characteristic $\chi(F)$ of the fixed point set $F = D^G$ for smooth actions of G on disks D. As a result, a closed smooth manifold F is (diffeomorphic to) the fixed point set of a smooth action of G on a disk if and only if F is (diffeomorphic to) the fixed point set of a smooth action of G on a Euclidean space (see Theorem 11).

Consider the reduced K-theory groups $\widetilde{KO}(F)$, $\widetilde{K}(F)$, and $\widetilde{KSp}(F)$ defined by using the real, complex, and quaternion vector bundles over F. Moreover, consider the maps

$$\widetilde{KO}(F) \xrightarrow{c_{\mathbb{R}}} \widetilde{K}(F) \xrightarrow{q_{\mathbb{C}}} \widetilde{KSp}(F) \text{ and } \widetilde{KSp}(F) \xrightarrow{c_{\mathbb{H}}} \widetilde{K}(F) \xrightarrow{r_{\mathbb{C}}} \widetilde{KO}(F)$$

where $c_{\mathbb{R}}$ and $q_{\mathbb{C}}$ are the induction (complexification and quaternization) maps and $c_{\mathbb{H}}$ and $r_{\mathbb{C}}$ are the forgetful (complexification and realification) maps.

In [36, Theorem 0.2 and Lemma 3.1], the class of finite groups G not of prime power order is divided into six mutually disjoint classes defined by using some G-modules, as well as by using only group theoretic terms. We recall the latter description and we adopt the following definitions. An element $g \in G$ is called *real* if g is conjugate to its inverse g^{-1}. Let p and q be two distinct primes. Then an element $g \in G$ is called a *pq-element* if g is of order pq. Moreover, we say that G has a *pq-dihedral subquotient* if G has two subgroups H and N such that $N \trianglelefteq H$ and H/N is isomorphic to the dihedral group of order $2pq$. Recall that G_2 denotes a Sylow 2-subgroup of G.

We define six mutually disjoint classes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ of finite groups G by assuming that G is not of prime power order and the following holds:

$\mathcal{A}$: G has a pq-dihedral subquotient, and thus G has a real pq-element.

$\mathcal{B}$: G has no pq-dihedral subquotient and G has a real pq-element.

$\mathcal{C}$: G has no real pq-element, G has a pq-element, and $G_2 \ntrianglelefteq G$.

$\mathcal{D}$: G has no real pq-element, G has a pq-element, and $G_2 \trianglelefteq G$.

$\mathcal{E}$: G has no pq-element and $G_2 \ntrianglelefteq G$.

$\mathcal{F}$: G has no pq-element and $G_2 \trianglelefteq G$.

If $G \in \mathcal{A} \cup \mathcal{B}$, then G has a real pq-element, and thus $G_2 \ntrianglelefteq G$. So, for a finite group G not of prime power order, $G_2 \trianglelefteq G$ if and only if $G \in \mathcal{D} \cup \mathcal{F}$. Hence, if G is perfect, $G \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{E}$.

Theorem 12 ([36]) *Let G be a finite group not of prime power order, and let $\mathcal{L} = \{G\}$. Let F be a compact smooth manifold with $\chi(F) \equiv 1 \pmod{n_G}$. Then the following three claims are equivalent.*

(1) F is the fixed point set of a smooth action of G on a disk.
(2) There exists a real $\mathcal{L}$-free G-vector bundle ν over F with $\mathcal{O}\ell(F, \nu) = 0$.
(3) For the class $[\tau_F] \in \widetilde{KO}(F)$ of the tangent bundle τ_F of F:

$G \in \mathcal{A}$: *there is no restriction on* $[\tau_F]$.

$G \in \mathcal{B}$: $c_{\mathbb{R}}([\tau_F]) \in c_{\mathbb{H}}(\widetilde{KSp}(F)) + \mathrm{tor}(\widetilde{K}(F))$.

$G \in \mathcal{C}$: $[\tau_F] \in r_{\mathbb{C}}(\widetilde{K}(F)) + \mathrm{tor}(\widetilde{KO}(F))$.

$G \in \mathcal{D}$: $[\tau_F] \in r_{\mathbb{C}}(\widetilde{K}(F))$ *(i.e., F is stably complex).*

$G \in \mathcal{E}$: $[\tau_F] \in \mathrm{tor}(\widetilde{KO}(F))$.

$G \in \mathcal{F}$: $[\tau_F] \in r_{\mathbb{C}}(\mathrm{tor}(\widetilde{K}(F)))$.

By [36], a result similar to Theorem 12 holds also for smooth actions of G on Euclidean spaces. In the corresponding version of Theorem 12, F is a smooth manifold with $\partial F = \varnothing$, and for $A = \widetilde{KO}(F)$ or $\widetilde{K}(F)$, the subgroup $\mathrm{tor}(A)$ of torsion elements in A is replaced by the subgroup $\mathrm{qdiv}(A)$ of quasidivisible elements in A. Recall that $\mathrm{qdiv}(A) = \mathrm{tor}(A)$ when A is finitely generated.

Oliver's results yield the following generalization of [36, Theorem 0.4] which should be compared with Theorem 3 in this article.

Theorem 13 ([36]) *Let G be a finite group not of prime power order, and let $\mathcal{L} = \{G\}$. Let F be a smooth manifold whose connected components $F_1, \dots, F_k$ are all stably parallelizable. Assume that F is compact and $\chi(F) \equiv 1 \pmod{n_G}$ (resp., $\partial F = \varnothing$). Let $V_1, \dots, V_k$ be real G-modules with $\dim V_i^G = \dim F_i$ for $1 \leq i \leq k$. Then the following two claims are equivalent.*

(1) F is the fixed point set of a smooth action of G on a disk (resp., Euclidean space) M such that $T_{x_i}(M) \cong V_i \oplus W$ for $x_i \in F_i$, $1 \leq i \leq k$, and some real $\mathcal{L}$-free G-module W.

(2) The G-modules V_i and V_j are $\mathcal{P}$-matched for $1 \leq i, j \leq k$.

For a finite group G, a real G-module V is called *G-oriented* if V^H is oriented for each $H \leq G$, and the transformation $g : V^H \to V^H$, $x \mapsto gx$ is orientation preserving for any $g \in N_G(H)$, the normalizer of H in G. A real G-module V is called *$\mathcal{P}$-oriented* if V^P is oriented for each $P \in \mathcal{P}(G)$, and $g : V^P \to V^P$ is orientation preserving for any $g \in N_G(P)$. A real G-vector bundle ν over a space with the trivial action of G is called *G-oriented* (resp., *$\mathcal{P}$-oriented*) if each fiber of ν is G-oriented (resp., *$\mathcal{P}$-oriented*) as a real G-module. Note that if V is the realification of a complex G-module, then V is G-oriented.

Let G be a finite group. For any connected oriented smooth G-manifold X, the *orientation homomorphism*

$$w_X : G \to \{\pm 1\}$$

is defined by setting $w_X(g) = 1$ if $g : X \to X$ is orientation preserving, and $w_X(g) = -1$ if $g : X \to X$ is orientation reversing.

Lemma 14 *Let G be a finite group. Let U and V be two real G-modules. Let E be a $\mathbb{Z}$-acyclic smooth G-manifold such that the tangent G-modules*

$$T_x(E) \cong U \oplus W \quad \text{and} \quad T_y(E) \cong V \oplus W$$

for some $x, y \in E^G$ and some real G-module W. Then, for each $P \in \mathcal{P}(G)$, the orientation homomorphisms

$$w_{U^P} : N_G(P)/P \to \{\pm 1\} \quad \text{and} \quad w_{V^P} : N_G(P)/P \to \{\pm 1\}$$

of the $N_G(P)/P$-modules U^P and V^P coincide, $w_{U^P} = w_{V^P}$.

PROOF. By Proposition 8, for each $P \in \mathcal{P}(G)$, the manifold E^P is stably complex, and thus E^P is orientable. As E^P is connected by Smith theory, the orientation homomorphism $w_{E^P} : N_G(P)/P \to \{\pm 1\}$ is well-defined. Clearly, $w_{(U \oplus W)^P} = w_{E^P} = w_{(V \oplus W)^P}$, and thus $w_{U^P} = w_{V^P}$. $\square$

Lemma 15 *Let G be a finite group and let U and V be two real $\mathcal{P}$-matched G-modules. Then the real G-module $U \oplus V$ is $\mathcal{P}$-oriented.*

PROOF. If G is of prime power order, the result is trivial. If G is not of prime power order, choose two connected stably parallelizable smooth manifolds M and N without boundary, such that $\dim M = \dim U^G$ and $\dim N = \dim V^G$. Set $F = M \sqcup N$, the disjoint union of M and N. As U and V are $\mathcal{P}$-matched, Theorem 13 asserts that F is the fixed point set of a smooth action of G on a Euclidean space E such that $T_x(E) \cong U \oplus W$ and $T_y(E) \cong V \oplus W$ for $x \in M$ and $y \in N$, and some real G-module W. Thus, by Lemma 14, $w_{U^P} = w_{V^P}$ for each $P \in \mathcal{P}(G)$. Hence, $w_{(U \oplus V)^P} : N_G(P)/P \to \{\pm 1\}$ is the trivial homomorphism, which means that $U \oplus V$ is $\mathcal{P}$-oriented. $\square$

Note that if V is a real $\mathcal{P}$-oriented G-module, then for any trivial G-submodule $M \subset V^G \subset V$, the orthogonal complement $V - M$ of M in V is $\mathcal{P}$-oriented, because $w_{V^P - M} = w_{V^P}$ for each $P \in \mathcal{P}(G)$.

Proposition 16 *Let G be a finite group. Let F be a smooth manifold and let ν be a real G-vector bundle over F with $\mathcal{O}\ell(F, \nu) = 0$. Let N be the real G-module determined on the fiber of ν over a point in F. Then the G-vector bundle $\tau_F \oplus \nu \oplus \varepsilon_F^N$ is $\mathcal{P}$-oriented and $\mathcal{O}\ell(F, \nu \oplus \varepsilon_F^N) = 0$.*

PROOF. For each connected component F_i of F $(i = 1, 2, \dots)$, let M_i be the trivial G-module with $\dim M_i = \dim F_i$ and let N_i be the real G-module determined on the fiber of ν over a chosen point in F_i. As $\mathcal{O}\ell(F, \nu) = 0$, the G-modules $M_i \oplus N_i$ and $M_j \oplus N_j$ are $\mathcal{P}$-matched for all i and j. Clearly, the G-module N is isomorphic to N_j for some j. Set $M = M_j$. By Lemma 15, $M_i \oplus N_i \oplus M \oplus N$ is $\mathcal{P}$-oriented, and thus so is $M_i \oplus N_i \oplus N$, proving that $\tau_F \oplus \nu \oplus \varepsilon_F^N$ is $\mathcal{P}$-oriented. As the Oliver obstruction remains unchanged under addition of product G-vector bundles, $\mathcal{O}\ell(F, \nu \oplus \varepsilon_F^N) = \mathcal{O}\ell(F, \nu) = 0$. $\square$

3 Equivariant thickening and surgery

Let G be a finite group. Henceforth, we denote by $\mathcal{S}(G)$ the family of all subgroups of G. Moreover, for any G-space X, we denote by $\mathcal{F}_{\mathrm{iso}}(G;X)$ the family of the isotropy subgroups G_x at points $x \in X$. We have already used the notation ε_X^V for the product G-vector bundle over X whose fiber is a real G-module V. When we write $\varepsilon_X^{k\mathbb{R}}$ for some integer $k \geq 0$, we consider the trivial action of G on $k\,\mathbb{R}$, where $k\,\mathbb{R} = \{0\}$ for $k = 0$, and $k\,\mathbb{R} = \mathbb{R} \oplus \cdots \oplus \mathbb{R}$ (k times) for $k \geq 1$.

Following [25], for a finite Oliver group G, we consider the real G-module

$$V(G) = (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_{p \,|\, |G|} (\mathbb{R}[G]^{O^p(G)} - \mathbb{R})$$

where each fixed point set $\mathbb{R}[G]^{O^p(G)}$ has the canonical action of G, and G acts trivially on the subtracted summands $\mathbb{R}$. According to [25],

$$\mathcal{F}_{\mathrm{iso}}(G; V(G) \smallsetminus \{0\}) = \mathcal{S}(G) \smallsetminus \mathcal{L}(G).$$

In particular, the G-module $V(G)$ is $\mathcal{L}$-free.

For constructions of smooth actions on disks, we use the following version of equivariant thickening developed in [38] and [39] (cf. [31, Theorem 3.1]).

Theorem 17 ([31]) *Let G be a finite Oliver group. Let M be a compact smooth G-manifold, let ν be a real $\mathcal{L}$-free G-vector bundle over M, let X be a finite contractible G-CW complex, and let ξ be a real G-vector bundle over X such that the following three conditions hold.*

(1) $X \supset M$ *as a G-invariant subcomplex.*
(2) $\xi|_M \cong \tau_M \oplus \nu \oplus \varepsilon_M^{\ell V(G)} \oplus \varepsilon_M^{k\mathbb{R}}$ *for some integers $k \geq 0$ and $\ell \geq 1$.*
(3) $\mathcal{F}_{\mathrm{iso}}(G; X \smallsetminus M) \subset \mathcal{S}(G) \smallsetminus \mathcal{L}(G)$.

If the integer ℓ is sufficiently large, then there exists a smooth action of G on a disk D such that the following three conclusions hold.

(1) $D \supset M$ *as a G-invariant submanifold, and* $\nu_{M \subset D} \cong \nu \oplus \varepsilon^{\ell V(G)}$.
(2) $\mathcal{F}_{\mathrm{iso}}(G; D \smallsetminus M) \subset \mathcal{S}(G) \smallsetminus \mathcal{L}(G)$, *and thus $D^H = M^H$ for each $H \in \mathcal{L}(G)$.*
(3) $D \supset X$ *as a G-invariant subcomplex, and there exists a G-deformation retraction $f : D \to X$ such that $\tau_D \oplus \varepsilon_D^{k\mathbb{R}} \cong f^*(\xi)$.*

Equivariant surgery is a poweful tool for constructing smooth actions of G on closed smooth manifolds, in particular, on spheres for finite groups G (see, e.g., [7]–[11], [15], [16], [42]–[45]). Petrie's equivariant surgery program (announced in [42] and [43]) has been elaborated in the articles [4], [5], [24]–[29]. As a result, so called "deleting–inserting theorems" were obtained in [24, Theorem 2.2] for any finite nonsolvable group G, and in [28, Theorems 0.1 and 4.1] for any finite Oliver group G. These theorems allow us to modify a suitable smooth action of G on a sphere S (resp., disk D) with fixed point set F so that for the resulting new action of G on S (resp., D), the fixed point set is obtained from F by deleting or inserting a number of connected components of F.

In Theorem 36, we discuss a generalization of [28, Theorem 0.1] which yields the result described in Theorem 18. First, for a finite group G, we denote by $\mathcal{PC}(G)$ the family of pseudocyclic subgroups of G (i.e., $H \in \mathcal{PC}(G)$ if and only if H/P is cyclic for some $P \trianglelefteq H$ with $P \in \mathcal{P}(G)$). Clearly, $\mathcal{P}(G) \subset \mathcal{PC}(G)$. By [25], $\mathcal{PC}(G) \cap \mathcal{L}(G) = \varnothing$ for any finite Oliver group G.

Let M be a smooth G-manifold, let F be a union of connected components of the fixed point set M^G, and let F' be the complement of F in M^G. Denote by

$$\mathcal{L}(M, F, F')$$

the union of the connected components of M^H intersecting both F and F', where H ranges over all large subgroups of G. If $\nu_{F \subset M}$ or $\nu_{F' \subset M}$ is $\mathcal{L}$-free, then $\mathcal{L}(M, F, F') = \varnothing$ by the Equivariant Tubular Neighborhood Theorem (see, e.g., [6, Theorem 2.2, p. 306] or [21, Theorem 4.8, p. 178]).

For any point $x \in M^G$, the tangent space $T_x(M)$ is considered as a real G-module by taking the derivatives (at the point x) of the transformations $g : M \to M$, $z \mapsto gz$ for all $g \in G$. We refer to this G-module $T_x(M)$ as to the *tangent G-module at x.*

Theorem 18 (cf. Theorem 36) *Let G be a finite Oliver group acting smoothly on a homotopy sphere Σ. Let F be a union of connected components of Σ^G. Assume that the following five conditions hold.*

(1) $\dim \Sigma^P > 2 \dim \Sigma^H$ *for all subgroups $P < H \leq G$ with $P \in \mathcal{P}(G)$.*
(2) $\dim \Sigma^P \geq 5$ *and* $\dim \Sigma^{=H} \geq 2$ *for any $P \in \mathcal{P}(G)$ and $H \in \mathcal{PC}(G)$.*
(3) Σ^P is simply connected for any $P \in \mathcal{P}(G)$.
(4) The tangent G-module $T_x(\Sigma)$ is $\mathcal{P}$-oriented for some $x \in F$.
(5) $\mathcal{L}(\Sigma, F, F') = \varnothing$ where F' is the complement of F in Σ^G.

Then there exists a smooth action of G on a sphere S of dimension $n = \dim \Sigma$, such that $S^G = F$ and $\nu_{F \subset S} \cong \nu_{F \subset \Sigma}$. Moreover, S^P is simply connected and $\dim S^P = \dim \Sigma^P$ *for each $P \in \mathcal{P}(G)$.*

In order to construct a smooth action of G on a sphere S with $S^G = F$ for a given closed smooth manifold F, we proceed as follows. First, by using Theorem 17, we construct a smooth action of G on a disk D with $D^G = F$. By doubling of D, we obtain a smooth action of G on $S = D \cup_{\partial D} D$ with

$S^G = F \sqcup F'$ where $F' = F$. Now, we would like to delete F' from S by using Theorem 18. To do it, we need to arrange the action of G on D so that doubling leads to an action of G on S satisfying the conditions (1)–(5) in Theorem 18. In particular, the condition (5) is ensured as follows. By using Theorems 19 and 20, we arrange the action of G on D with $D^G = F$ so that $\nu_{F \subset D}$ is $\mathcal{L}$-free. Then, for the action of G on S obtained by doubling of D, $\nu_{F \subset S}$ and $\nu_{F' \subset S}$ are both $\mathcal{L}$-free, which shows that $\mathcal{L}(S, F, F') = \varnothing$.

4 Equivariant bundle extension and subtraction

Following Oliver [36], for a finite group G not of prime power order, consider the classifying space B_GO of real G-vector bundles as the infinite mapping cylinder of maps

$$B_GO(0) \to B_GO(r) \to B_GO(2r) \to B_GO(3r) \to \dots$$

where $r = |G|$ and for $n \geq 0$, $B_G(nr)$ is the classifying space of nr-dimensional real G-vector bundles and the map $B_GO(nr) \to B_GO((n+1)r)$ is stabilization by the real regular G-module $\mathbb{R}[G]$ (cf. [12, Theorem I.8.12]). Oliver [36] defines a G-space B_G^*O and a G-map $L_G : B_GO \to B_G^*O$, and proves that L_G is a (nonequivariant) homotopy equivalence [36, Definition 1.1 and Lemma 2.1] and shows how to construct G-maps f from a finite G-CW complex X into B_G^*O and how to lift f to B_GO [36, Propositions 1.3 and 2.3]. Then he shows how to construct G-vector bundles by combining a given G-vector bundle and a family of P-vector bundles (where $P \in \mathcal{P}(G)$) satisfying some compatibility (see [36, Theorem 2.4] for the details of Oliver's procedure).

In [30], we have described an equivariant wedge sum construction to obtain some refinements of Oliver's procedure. In particular, a corresponding G-CW complex X is constructed so that the fixed point set X^P is simply connected for each $P \in \mathcal{P}(G)$. The statements of [30, Theorems 1.3, 1.4, 5.1, 5.2] depend on a set T of primes $p \Big| |G|$. We restate the conclusion of [30, Theorem 5.2] in the special case where T is the set of all primes $p \Big| |G|$. First, we introduce the notion of L_G-system. So, a quintuple $\mathbb{X} = (X, Y, \varphi_X, f_X, h_X)$ is called an *L_G-system* if the following five conditions hold.

(1) X is a finite G-CW complex with base point $x_0 \in X^G$.
(2) Y is a finite G-CW complex with base point $y_0 \in Y^G$.
(3) $\varphi_X : X \to Y$ is a G-map with $\varphi(x_0) = y_0$.
(4) $f_X : X \to B_GO$ is a G-map.
(5) $h_X : X \times I \to B_G^*O$ is a G-homotopy from $L_G \circ f_X$ to $c_X = c_Y \circ \varphi_X$, where $c_Y : Y \to B_G^*O$ is the constant map into the point $L_G(f_X(x_0))$.

The notion of L_G-system was used for the first time in [30] under the name of (B_GO, B_G^*O)-system. For an L_G-system $\mathbb{X} = (X, Y, \varphi_X, f_X, h_X)$, one obtains the following G-homotopically commutative diagram

$$\begin{tikzcd} X \arrow[r, "f_X"] \arrow[d, "\varphi_X"'] & B_GO \arrow[d, "L_G"] \\ Y \arrow[r, "c_Y"] & B_G^*O. \end{tikzcd}$$

Following [31], we say that an L_G-system $\mathbb{A} = (A, B, \varphi_A, f_A, h_A)$ *extends to* an L_G-system $\mathbb{X} = (X, Y, \varphi_X, f_X, h_X)$ if $X \supset A$ and $Y \supset B$ as G-invariant subcomplexes, $\varphi_X|_A = \varphi_A$, $f_X|_A = f_A$, and $h_X|_{A\times I} = h_A$. Now, we are ready to restate [30, Theorem 5.2] in the case T is the set of all primes $p \Big| |G|$.

Theorem 19 ([30]). *Let G be a finite Oliver group. Let A be a finite G-CW complex such that fixed point set $F = A^G$ is nonempty and the fixed point set A^P is simply connected for each $P \in \mathcal{P}(G)$ with $P \neq \{e\}$. Then any L_G-system $\mathbb{A} = (A, A, id_A, f_A, h_A)$ extends to an L_G-system $\mathbb{X} = (X, Y, \varphi_X, f_X, h_X)$ such that X is contractible, and such that $X^G = F$ and X^P is simply connected for each $P \in \mathcal{P}(G)$, and for each $H \in \mathcal{S}(G) \smallsetminus \mathcal{PC}(G)$, $X^H \smallsetminus F$ is a discrete space and $f_X|_{X^H \smallsetminus F}$ is the constant map into the point $f(x_0) \in B_GO$.*

By using Theorem 19, we perform extensions of G-vector bundles. Therefore, the resulting G-vector bundle over X contains ε_X^V as a direct summand, where $\mathcal{F}_{\text{iso}}(G; V \smallsetminus \{0\}) = \mathcal{S}(G)$. To ensure that $\mathcal{F}_{\text{iso}}(G; V \smallsetminus \{0\}) \subset \mathcal{S}(G) \smallsetminus \mathcal{L}(G)$, we apply the following version of the equivariant bundle subtraction procedure described in [31, Theorem 2.2] (see also [31, Proposition 2.3]).

Theorem 20 ([31]) *Let G be a finite group. Let (X, A) be a pair consisting of finite G-CW complexes X and A such that $X \supset A$ as a G-invariant subcomplex and $\mathcal{F}_{\text{iso}}(G; X \smallsetminus A) \subset \mathcal{S}(G) \smallsetminus \mathcal{L}(G)$. Let V be a real G-module. Let ξ and η be G-vector bundles over X and A, respectively, such that*

$$\xi|_A \cong \eta \oplus \varepsilon_A^V \oplus \varepsilon_A^{\ell V(G)} \quad \textit{for some integer} \quad \ell \geq 1.$$

If the integer ℓ is sufficiently large, then ξ contains a G-subbundle $\theta \cong \varepsilon_X^V$ with G-orthogonal complement $\xi - \theta$ such that $(\xi - \theta)|_A \cong \eta \oplus \varepsilon_A^{\ell V(G)}$.

5 Constructions of group actions on disks

The following theorem goes back to [31, Theorem 0.3 and Section 3], and its proof makes use of the methods in [36] (cf. Theorems 11 and 12 in this article) and the procedure in [31, Theorem 2.2] (cf. Theorem 20 in this article).

Theorem 21 *Let G be a finite Oliver group. Let F be a compact smooth manifold and let ν be a real $\mathcal{L}$-free G-vector bundle over F with $\mathcal{O}\ell(F,\nu)=0$. Then, for any sufficiently large integer ℓ, there exists a smooth action of G on a disk D such that $D^G = F$ and $\nu_{F\subset D} \cong \nu \oplus \varepsilon_F^{\ell V(G)}$.*

Now, we wish to obtain stronger versions of Theorem 21 under the additional conditions that the manifold F is simply connected or F is both connected and stably parallelizable. In these cases, we prove that the action of G on D with $D^G = F$ can be constructed in such a way that D^P is simply connected for each $P \in \mathcal{P}(G)$. Note that in the case F is stably parallelizable, $\mathcal{O}\ell(F,\nu)=0$ for $\nu = \varepsilon_F^V$ and any real G-module V.

Theorem 22 *Let G be a finite Oliver group. Let F be a compact connected smooth manifold. Assume that F is simply connected. Let ν be a real $\mathcal{L}$-free G-vector bundle over F with $\mathcal{O}\ell(F,\nu)=0$. Then, for any sufficiently large integer ℓ, there exists a smooth action of G on a disk D such that $D^G = F$ and D^P is simply connected for each $P \in \mathcal{P}(G)$, and $\nu_{F\subset D} \cong \nu \oplus \varepsilon_F^{\ell V(G)}$.*

PROOF. Let $f : F \to B_G O$ be the classifying map of the Whitney sum $\tau_F \oplus \nu$. Choose a point $x_0 \in F$ and set $b_0 = L_G(f(x_0))$. As $\mathcal{O}\ell(F,\nu)=0$, the composition $L_G \circ f : F \to B_G^* O$ is G-homotopic to the constant map into the point b_0 via a G-homotopy $h : F \times I \to B_G^* O$ (cf. [36]). As F is simply connected, Theorem 19 asserts that the L_G-system $\mathbb{F} = (F, F, id_F, f, h)$ extends to an L_G-system $\mathbb{X} = (X, Y, \varphi_X, f_X, h_X)$ such that X is contractible, $X^G = F$, X^P is simply connected for each $P \in \mathcal{P}(G)$, $X^H \smallsetminus F$ is a discrete space for each $H \in \mathcal{N}(G) = \mathcal{S}(G) \smallsetminus \mathcal{PC}(G)$, and the G-map $f_X : X \to B_G O$ when restricted on the discrete subspace B of X,

$$B = \bigcup_{H \in \mathcal{N}(G)} X^H \smallsetminus F,$$

is the constant map of B into the point $f(x_0) \in B_G O$. Let V be the real G-module determined on the fiber of $\tau_F \oplus \nu$ over the point x_0. Set $A = F \cup B$. Let η be the G-vector bundle over A defined by setting $\eta|_F = \tau_F \oplus \nu$ and $\eta|_B = \varepsilon_B^V$. Clearly, the G-map f_X yields a real G-vector bundle ξ over X such that $\xi|_A \cong \eta \oplus \varepsilon_F^{k\mathbb{R}[G]}$ for an integer $k \geq 0$. Thus, for any integer $\ell \geq 0$,

$$(\xi \oplus \varepsilon_X^{\ell V(G)})|_A \cong \eta \oplus \varepsilon_A^{k\mathbb{R}[G]} \oplus \varepsilon_A^{\ell V(G)}.$$

As $\mathcal{F}_{\mathrm{iso}}(G; X \smallsetminus A) \subset \mathcal{PC}(G) \subset \mathcal{S}(G) \smallsetminus \mathcal{L}(G)$, we may apply Theorem 20 to conclude that if ℓ is sufficiently large, $\xi \oplus \varepsilon_X^{\ell V(G)}$ contains a G-subbundle θ such that $\theta \cong \varepsilon_X^{k\mathbb{R}[G]}$ and $(\xi \oplus \varepsilon_X^{\ell V(G)} - \theta)|_A \cong \eta \oplus \varepsilon_A^{\ell V(G)}$. Therefore,

$$(\xi \oplus \varepsilon_X^{\ell V(G)} - \theta)|_F \cong \tau_F \oplus \nu \oplus \varepsilon_F^{\ell V(G)}$$

and

$$(\xi \oplus \varepsilon_X^{\ell V(G)} - \theta)|_B \cong \varepsilon_B^V \oplus \varepsilon_B^{\ell V(G)}.$$

Let C be a disk with $\dim C = \dim F$. Take $E = B \times C$ with the diagonal action of G, where G acts trivially on C. Then E is a smooth G-manifold such that $E \supset B$ as a strong G-deformation retract (each point $b \in B$ is identified with $(b, c_0) \in B \times C$, where c_0 is the origin of C). Consider the union $X \cup_B E$ of X and E along B and take the obvious strong G-deformation retraction $r : X \cup_B E \to X$. Let M be the disjoint union of F and E. By construction, M is a smooth G-manifold with $M^G = F$ and $X \cup_B E$ is a finite contractible G-CW complex containing M as a G-invariant subcomplex such that

$$\mathcal{F}_{\text{iso}}(G; (X \cup_B E) \smallsetminus M) \ \subset \ \mathcal{PC}(G) \ \subset \ \mathcal{S}(G) \smallsetminus \mathcal{L}(G).$$

In particular, $(X \cup_B E)^G = F$. Set $\overline{\xi} = r^*(\xi \oplus \varepsilon_X^{\ell V(G)} - \theta)$. Then

$$\overline{\xi}|_F \cong \tau_F \oplus \nu \oplus \varepsilon_F^{\ell V(G)} \quad \text{and} \quad \overline{\xi}|_E \cong \tau_E \oplus \varepsilon_E^{V - V^G} \oplus \varepsilon_E^{\ell V(G)}.$$

Let $\overline{\nu}$ be the G-vector bundle over M with $\overline{\nu}|_F = \nu$ and $\overline{\nu}|_E = \varepsilon_E^{V-V^G}$. Then $\overline{\xi}|_M \cong \tau_M \oplus \overline{\nu} \oplus \varepsilon_M^{\ell V(G)}$. As the conditions (1)–(3) in Theorem 17 all hold for M, $\overline{\nu}$, $X \cup_B E$, and $\overline{\xi}$, we obtain the corresponding conclusions (1)–(3) in Theorem 17, provided the integer ℓ is sufficiently large. Therefore, there exists a smooth action of G on a disk D such that $D \supset M$ as a G-invariant submanifold with $\nu_{M \subset D} \cong \overline{\nu} \oplus \varepsilon_M^{\ell V(G)}$, and the following holds:

$$\mathcal{F}_{\text{iso}}(G; D \smallsetminus M) \subset \mathcal{PC}(G) \subset \mathcal{S}(G) \smallsetminus \mathcal{L}(G),$$

$D^G = M^G = F$, $\nu_{F \subset D} \cong \nu \oplus \varepsilon_F^{\ell V(G)}$, $D \supset X$ as a G-invariant subcomplex, and there exists a strong G-deformation retraction $f : D \to X$ such that $f^*(\overline{\xi}) \cong \tau_D$. In particular, D^H and X^H have the same homotopy type for each $H \in \mathcal{S}(G)$. As X^P is simply connected for each $P \in \mathcal{P}(G)$, so is D^P. □

Theorem 23 *Let G be a finite Oliver group. Let F be a compact connected smooth manifold. Assume also that F is stably parallelizable. Let V be a real G-module with* $\dim V^H = \dim F$ *for each $H \in \mathcal{L}(G)$. Then, for any sufficiently large integer ℓ, there exists a smooth action of G on a disk D such that $D^G = F$ and D^P is simply connected for each $P \in \mathcal{P}(G)$, and $\nu_{F \subset D} \cong \varepsilon_F^{V-V^G} \oplus \varepsilon_F^{\ell V(G)}$ where $V - V^G$ is the G-orthogonal complement of V^G in V.*

PROOF. As $n_G = 1$ and F is compact, it follows from [31, Lemma 3.6] that there exists a finite contractible G-CW complex X such that $X^G = F$, X^P is simply connected for each $P \in \mathcal{P}(G)$, and $X^H \smallsetminus F$ is a discrete space for each $H \in \mathcal{N}(G) = \mathcal{S}(G) \smallsetminus \mathcal{CP}(G)$. Define B and E, and consider $X \cup_B E$ as in the proof of Theorem 22. Set $d = \dim F = \dim E$. Let $M = F \sqcup E$, the disjoint union of F and E. As F and E are stably parallelizable, so is M. Therefore, $\tau_M \oplus \varepsilon_M^{\mathbb{R}} \cong \varepsilon_M^{(d+1)\mathbb{R}}$. Consider the G-vector bundles

$$\xi = \varepsilon_{X\cup_B E}^{(d+1)\mathbb{R}} \oplus \varepsilon_{X\cup_B E}^{V\oplus \ell V(G)} \quad \text{and} \quad \nu = \varepsilon_M^V.$$

As $X\cup_B E \supset M$ and the family $\mathcal{F}_{\mathrm{iso}}(G; (X\cup_B E)\smallsetminus M) \subset \mathcal{S}(G)\smallsetminus\mathcal{L}(G)$, and the bundle $\xi|_M \cong \tau_M \oplus \nu \oplus \varepsilon_M^{\ell V(G)} \oplus \varepsilon_M^{\mathbb{R}}$, the conditions (1)–(3) in Theorem 17 all hold for M, ν, $X\cup_B E$, and ξ. Thus the result follows by applying Theorem 17 (cf. the arguments in the proof of Theorem 22). $\square$

6 Constructions of group actions on spheres

Let G be a finite group. For two subgroups P and H of G, the pair (P, H) is called *proper* if $P \in \mathcal{P}(G)$ and P is a proper subgroup of H. A proper pair (P, H) of subgroups of G is called *odd* if $|H : P| = |H(O^2(G)) : P(O^2(G))| = 2$ and $P(O^p(G)) = G$ for all odd primes p. A proper pair (P, H) of subgroups of G is called *even* if (P, H) is not odd.

Following [32], for a real G-module V and a proper pair (P, H) of subgroups of G, we set $d_V(P, H) = \dim V^P - 2\dim V^H$. By [25], the following lemma holds for the real G-module $V(G) = (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_{p\,|\,|G|}(\mathbb{R}[G]^{O^p(G)} - \mathbb{R})$.

Lemma 24 ([25]) *For any proper pair (P, H) of subgroups of G,*

(1) $d_{V(G)}(P, H) = 0$ when (P, H) is odd, and
(2) $d_{V(G)}(P, H) > 0$ when (P, H) is even.

A real G-module V is called a *gap G-module* if $d_V(P, H) > 0$ for each proper pair (P, H) of subgroups of G. Thus, a finite group G is a gap group if and only if $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ and G has a real $\mathcal{L}$-free gap G-module.

Let G be a finite Oliver group. Then $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ and $V(G)$ is $\mathcal{L}$-free (cf. [25]). By [32], if $O^p(G) \neq G$ and $O^q(G) \neq G$ for two distinct odd primes p and q, or $O^2(G) = G$, then each proper pair (P, H) of subgroups of G is even. Hence, $V(G)$ is a gap G-module by Lemma 24, and thus G is a gap group.

Lemma 25 *Let G be a finite Oliver gap group and let V be a real $\mathcal{L}$-free gap G-module. Then $2V \oplus 2V(G)$ is a real G-oriented $\mathcal{L}$-free gap G-module.*

PROOF. Clearly, $2V \oplus 2V(G)$ is both G-oriented and $\mathcal{L}$-free. For each proper pair (P, H) of subgroups of G, $d_V(P, H) > 0$ as V is a gap G-module, and

$d_{V(G)}(P,H) \geq 0$ by Lemma 24. Therefore

$$d_{2V\oplus 2V(G)}(P,H) = 2d_V(P,H) + 2d_{V(G)}(P,H) > 0,$$

proving that $2V \oplus 2V(G)$ is a gap G-module. □

Proposition 26 *Let G be a finite Oliver group with a real $\mathcal{P}$-oriented $\mathcal{L}$-free gap G-module V which contains $V(G)$ as a direct summand. Let F be a closed smooth manifold whose connected components are all simply connected. Let ν be a real $\mathcal{P}$-oriented $\mathcal{L}$-free G-vector bundle over F with $\mathcal{O}\ell(F,\nu)=0$. Then, for any sufficiently large integer ℓ, there exists a smooth $\mathcal{P}$-typical action of G on a sphere S such that $S^G = F$ and $\nu_{F\subset S} \cong \nu \oplus \varepsilon_F^{\ell V}$.*

PROOF. As $\mathcal{F}_{\text{iso}}(G; V(G) \smallsetminus \{0\}) = \mathcal{S}(G) \smallsetminus \mathcal{L}(G)$ and $\mathcal{PC}(G) \cap \mathcal{L}(G) = \varnothing$ by [25], $V(G)$ is $\mathcal{L}$-free and $\dim(\ell\, V(G))^{=H} \geq \ell$ for each $H \in \mathcal{S}(G) \smallsetminus \mathcal{L}(G)$ and each integer $\ell \geq 1$. By assumption, V is $\mathcal{P}$-oriented, $\mathcal{L}$-free, gap, and V contains $V(G)$ as a direct summand.

Let $F_1, \ldots, F_k$ be the connected components of F. For $i = 1, \ldots, k$ and an integer $\ell \geq 1$, set $\nu_i = \nu|_{F_i} \oplus \varepsilon_{F_i}^{\ell W}$ where $W = V - V(G)$, the G-orthogonal complement of $V(G)$ in V. Note that ν_i is $\mathcal{L}$-free because so are $\nu|_{F_i}$ and W. As F_i is simply connected and $\mathcal{O}\ell(F_i,\nu_i)=0$, it follows from Theorem 22 that if the integer ℓ is sufficiently large, there exists a smooth action of G on a disk D_i with $\dim D_i = \dim(\tau_F \oplus \nu) + \ell \dim V$, such that $D_i^G = F_i$ and D_i^P is simply connected for each $P \in \mathcal{P}(G)$, and

$$\nu_{F_i\subset D_i} \cong \nu_i \oplus \varepsilon_{F_i}^{\ell V(G)} \cong \nu|_{F_i} \oplus \varepsilon_{F_i}^{\ell(W\oplus V(G))} \cong \nu|_{F_i} \oplus \varepsilon_{F_i}^{\ell V}.$$

As $\nu|_{F_i}$ and V are both $\mathcal{P}$-oriented, the G-module $T_{x_i}(D_i)$ is $\mathcal{P}$-oriented for any $x_i \in F_i$. Let V_i be the G-module determined on the fiber of $\tau_F \oplus \nu$ over a chosen point in F_i. As V is a gap G-module, $d_V(P,H) > 0$ for each proper pair (P,H) of subgroups of G. The integer ℓ can be chosen so that for each proper pair (P,H) of subgroups of G, $\ell\, d_V(P,H) > -d_{V_i}(P,H)$ and thus

$$d_{V_i\oplus\ell V}(P,H) = d_{V_i}(P,H) + \ell\, d_V(P,H) > d_{V_i}(P,H) - d_{V_i}(P,H) = 0.$$

Therefore, $\dim D_i^P = \dim(V_i^P \oplus \ell\, V^P) > 2\dim(V_i^H \oplus \ell\, V^H) = 2\dim D_i^H$. Also, due to the properties of $V(G)$, the integer ℓ can be chosen so that for each $P \in \mathcal{P}(G)$ and $H \in \mathcal{PC}(G)$, $\dim D_i^P \geq 5$ and $\dim D_i^{=H} \geq 2$.

Let S_i be the equivariant double of D_i. Then $S_i^G = F_i \sqcup F_i'$, where $F_i' = F_i$, and $\nu_{F_i\subset S_i} \cong \nu_{F_i'\subset S_i} \cong \nu|_{F_i} \oplus \varepsilon_{F_i}^{\ell V}$. As $\nu|_{F_i}$ and V are both $\mathcal{L}$-free, so is $\nu|_{F_i} \oplus \varepsilon_{F_i}^{\ell V}$, and thus $\nu_{F_i\subset S_i}$ and $\nu_{F_i'\subset S_i}$ are both $\mathcal{L}$-free. As a result, the following five conditions hold for the resulting action of G on S_i.

(1) $\dim S_i^P > 2\dim S_i^H$ for all subgroups $P < H \leq G$ with $P \in \mathcal{P}(G)$.

(2) $\dim S_i^P \geq 5$ and $\dim S_i^{=H} \geq 2$ for any $P \in \mathcal{P}(G)$ and $H \in \mathcal{PC}(G)$.
(3) S_i^P is simply connected for any $P \in \mathcal{P}(G)$.
(4) The tangent G-module $T_{x_i}(S_i)$ is $\mathcal{P}$-oriented for any $x_i \in F_i$.
(5) $\mathcal{L}(S_i, F_i, F_i') = \varnothing$.

The conditions (1)–(5) allow us to delete F_i' from S_i by applying Theorem 18, i.e., we obtain a smooth action of G on S_i such that $S_i^G = F_i$ and S_i^P is simply connected for each $P \in \mathcal{P}(G)$, and $\nu_{F_i \subset S_i} \cong \nu|_{F_i} \oplus \varepsilon_{F_i}^{\ell V}$. In particular, $T_{x_i}(S_i) \cong d_i\mathbb{R} \oplus N_i \oplus \ell V$ where G acts trivially on $d_i\mathbb{R}$ for $d_i = \dim F_i$, and N_i is the real G-module determined on the fiber of ν over a point $x_i \in F_i$.

For each $i = 1, \ldots, k$, take a disk B_i with $\dim B_i = d_i$. Let ν_B be the real G-vector bundle over the disjoint union $B = B_1 \sqcup \cdots \sqcup B_k$ which restricts to $\varepsilon_{B_i}^{N_i \oplus \ell V}$ over B_i. As $\mathcal{O}\ell(F, \nu) = 0$, it follows that $\mathcal{O}\ell(B, \nu_B) = 0$. Hence, if the integer ℓ is sufficiently large, then by Theorem 21, there exists a smooth action of G on a disk D_0 with $\dim D_0 = \dim D_i$, such that

$$D_0^G = B_1 \sqcup \cdots \sqcup B_k \quad \text{and} \quad \nu_{B_i \subset D_0} \cong \varepsilon_{B_i}^{N_i \oplus \ell V} \quad \text{for} \quad i = 1, \ldots, k.$$

The equivariant double of D_0 is a sphere S_0 (with $\dim S_0 = \dim S_i$) upon which G acts smoothly in such a way that

$$S_0^G = A_1 \sqcup \cdots \sqcup A_k \quad \text{and} \quad \nu_{A_i \subset S_0} \cong \varepsilon_{A_i}^{N_i \oplus \ell V} \quad \text{for} \quad i = 1, \ldots, k,$$

where A_i is the sphere obtained by doubling of B_i. For each $i = 1, \ldots, k$, choose a point $a_i \in A_i \subset S_0$. Note that $T_{a_i}(S_0) \cong d_i\mathbb{R} \oplus N_i \oplus \ell V \cong T_{x_i}(S_i)$. Therefore, we can take the equivariant connected sum $S = S_0 \# S_1 \# \ldots \# S_k$ formed by connecting sufficiently small G-invariant spheres around the points $a_i \in A_i \subset S_0$ and $x_i \in F_i \subset S_i$. As a result, we obtain a smooth action of G on the sphere S such that $S^G = F_1 \sqcup \cdots \sqcup F_k = F$ and $\nu_{F \subset S} \cong \nu \oplus \varepsilon_F^{\ell V}$. Note that by our construction, $S^P \neq F$ for each $P \in \mathcal{P}(G)$, and thus the action of G on S is $\mathcal{P}$-typical. □

Theorem 27 *Let G be a finite Oliver gap group. Let F be a closed smooth manifold whose connected components are all simply connected. Let ν be a real $\mathcal{L}$-free G-vector bundle over F with $\mathcal{O}\ell(F, \nu) = 0$. Then there exists a smooth $\mathcal{P}$-typical action of G on a sphere S such that $S^G = F$ and $\nu_{F \subset S} \cong \nu \oplus \varepsilon_F^W$ for some real $\mathcal{L}$-free G-module W. If $\tau_F \oplus \nu$ is $\mathcal{P}$-oriented, W can be chosen to be the realification of a complex G-module.*

PROOF. As G is a gap group, Lemma 25 asserts that G has a real $\mathcal{P}$-oriented $\mathcal{L}$-free gap G-module V which contains $V(G)$ as a direct summand, and V can be chosen to be the realification of a complex G-module.

Let N be the real G-module determined on the fiber of ν over a point in F. By Proposition 16, the G-vector bundle $\tau_F \oplus \nu \oplus \varepsilon_F^N$ is $\mathcal{P}$-oriented. Clearly,

$\nu \oplus \varepsilon_F^N$ is $\mathcal{L}$-free and $\mathcal{O}\ell(F, \nu \oplus \varepsilon_F^N) = 0$ because $\mathcal{O}\ell(F, \nu) = 0$ by assumption. Set $W = N \oplus \ell V$ for an integer $\ell \geq 1$. If the integer ℓ is sufficiently large, then by Proposition 26, there exists a smooth $\mathcal{P}$-typical action of G on a sphere S such that

$$S^G = F \quad \text{and} \quad \nu_{F \subset S} \cong \nu \oplus \varepsilon_F^N \oplus \varepsilon_F^{\ell V} \cong \nu \oplus \varepsilon_F^W.$$

If $\tau_F \oplus \nu$ is $\mathcal{P}$-oriented, then by Proposition 26, there exists a smooth $\mathcal{P}$-typical action of G on a sphere S such that $S^G = F$ and $\nu_{F \subset S} \cong \nu \oplus \varepsilon_F^W$ for $W = \ell V$, provided the integer ℓ is sufficiently large. As V is the realification of a complex G-module, so is W. $\square$

Theorem 28 *Let G be a finite Oliver gap group. Let F be a closed smooth manifold whose connected components $F_1, \ldots, F_k$ are all stably parallelizable. Let $V_1, \ldots, V_k$ be real G-modules such that $\dim V_i^H = \dim F_i$ for all $H \in \mathcal{L}(G)$ and $1 \leq i \leq k$, and such that V_i and V_j are $\mathcal{P}$-matched for all $1 \leq i, j \leq k$. Then there exists a smooth $\mathcal{P}$-typical action of G on a sphere S such that $S^G = F$ and $T_{x_i}(S) \cong V_i \oplus W$ for all $x_i \in F_i$, $1 \leq i \leq k$, and some real $\mathcal{L}$-free G-module W. If each V_i is $\mathcal{P}$-oriented, W can be chosen to be the realification of a complex G-module.*

PROOF. Again, Lemma 25 asserts that G has a real $\mathcal{P}$-oriented $\mathcal{L}$-free gap G-module V which contains $V(G)$ as a direct summand, and V can be chosen to be the realification of a complex G-module.

For each $i = 1, \ldots, k$, set $N_i = V_i - V_i^G$, the G-orthogonal complements of V_i^G in V_i. Let ν be the G-vector bundle over F which restricts to $\varepsilon_{F_i}^{N_i}$ for $i = 1, \ldots, k$. As V_i and V_j are $\mathcal{P}$-matched for all $1 \leq i, j \leq k$, and each F_i is stably parallelizable, thus $\mathcal{O}\ell(F, \nu) = 0$. Moreover, by Proposition 16, the real $\mathcal{L}$-free G-vector bundle $\tau_F \oplus \nu \oplus \varepsilon_F^N$ is $\mathcal{P}$-oriented, where N is the real G-module determined on the fiber of ν over a chosen point in F. As in the proof of Theorem 27, set $W = N \oplus \ell V$ for an integer $\ell \geq 1$.

For each $i = 1, \ldots, k$, F_i is stably parallelizable, and thus Theorem 23 asserts that there exists a smooth action of G on a disk D_i such that $D_i^G = F_i$ and D_i^P is simply connected for each $P \in \mathcal{P}(G)$, and $\nu_{F_i \subset D_i} \cong \varepsilon_{F_i}^{N_i} \oplus \varepsilon_{F_i}^W$, provided the integer ℓ is sufficiently large. Clearly, $T_{x_i}(D_i) \cong V_i \oplus W$ for $x_i \in F_i$.

To complete the proof, we proceed as in the proof of Proposition 26. We take the equivariant double S_i of D_i, so that $S_i^G = F_i \sqcup F_i'$ for $F_i' = F_i$. Then, using Theorem 18, we delete F_i' from the sphere S_i to obtain a smooth action of G on S_i such that $S_i^G = F_i$ and $T_{x_i}(S_i) \cong V_i \oplus W$ for $x_i \in F_i$. Now, as in the proof of Proposition 26, we use Theorem 21 to obtain a smooth action of G on a disk D_0 such that

$$D_0^G = B_1 \sqcup \cdots \sqcup B_k \quad \text{and} \quad \nu_{B_i \subset D_0} \cong \varepsilon_{B_i}^{N_i \oplus W} \quad \text{for} \quad i = 1, \ldots, k,$$

where each B_i is a disk with $\dim B_i = \dim F_i$. Clearly, $T_{b_i}(D_0) \cong V_i \oplus W$ for all $b_i \in B_i$. Therefore, as in the proof of Proposition 26, we can take the equivarinat double S_0 of D_0, and then we can form the equivariant connected sum $S = S_0 \# S_1 \# \dots \# S_k$ to obtain a smooth $\mathcal{P}$-typical action of G on S such that $S^G = F_1 \sqcup \cdots \sqcup F_k = F$ and $T_{x_i}(S) \cong V_i \oplus W$ for all $x_i \in F_i$.

If each V_i is $\mathcal{P}$-oriented, then $\tau_F \oplus \nu$ is also $\mathcal{P}$-oriented, and thus instead of taking $W = N \oplus \ell V$, we can set $W = \ell V$ to ensure that W is the realification of a complex G-module. $\square$

Corollary 29 *Let G be a finite Oliver gap group. Let F be a closed smooth manifold. Assume that each connected component of F is simply connected or stably parallelizable. Let ν be a real $\mathcal{L}$-free G-vector bundle over F with $\mathcal{O}\ell(F, \nu) = 0$. Then F is the fixed point set of a smooth $\mathcal{P}$-typical action of G on a sphere S, and $\nu_{F \subset S}$ can be chosen to be the realification of a complex $\mathcal{L}$-free G-vector bundle, provided ν is complex.*

PROOF. Let $F_1, \dots, F_k$ be the connected components of F. Let $V_1, \dots, V_k$ be the real G-modules determined on the fibers of $\tau_F \oplus \nu$ over some points $x_i \in F_i$ for $i = 1, \dots, k$. As ν is $\mathcal{L}$-free, $\dim V_i^H = \dim F_i$ for all $H \in \mathcal{L}(G)$. Set $N_i = V_i - V_i^G$, the G-orthogonal complement of V_i^G in V_i.

Let $\overline{\nu}$ be the real G-vector bundle over F with $\overline{\nu}|_{F_i} = \nu|_{F_i}$ when F_i is simply connected, and $\overline{\nu}|_{F_i} = \varepsilon_{F_i}^{N_i}$ when F_i is not simply connected. Then $\overline{\nu}$ is $\mathcal{L}$-free, $\mathcal{O}\ell(F, \overline{\nu}) = 0$, and the G-modules V_i and V_j are $\mathcal{P}$-matched for $1 \le i, j \le k$. Now, by applying the arguments used to prove Theorems 27 and 28, we obtain smooth actions of G on spheres $S_1, \dots, S_k$ such that $S_i^G = F_i$ for $i = 1, \dots, k$, and a smooth action of G on a sphere S_0, which allow us to form the equivariant connected sum $S = S_0 \# S_1 \# \dots \# S_k$, and thus to obtain a smooth action of G on the sphere S such that $S^G = F$ and $\nu_{F \subset S} \cong \overline{\nu} \oplus \varepsilon_F^W$.

Note that if ν is the realification of a complex G-vector bundle, so is $\overline{\nu}$, and thus $\tau_F \oplus \overline{\nu}$ is $\mathcal{P}$-oriented. Hence, as in the proofs of Theorems 27 and 28, we can set $W = \ell V$ for the realification V of a complex G-module, proving that $\nu_{F \subset S} \cong \overline{\nu} \oplus \varepsilon_F^W$ is the realification of a complex G-vector bundle. $\square$

7 Special pairs of group modules

Let G be a finite group. By the definition given in the introduction, two real (resp., complex) G-modules U and V are $\mathcal{P}$-matched if U and V are isomorphic as real (resp., complex) P-modules for each $P \in \mathcal{P}(G)$.

In this article, a pair (U, V) of real (resp., complex) G-modules U and V is called *special* if U and V are $\mathcal{P}$-matched, the nontrivial summands $U - U^G$ and $V - V^G$ are $\mathcal{L}$-free, and $\dim V^G - \dim U^G = 1$, where the fixed point sets U^G and V^G are considered as real (resp., complex) vector spaces.

Lemma 30 *Let G be a finite group with a special pair of real (resp., complex) G-modules. Let F be a compact (resp., compact and stably complex) smooth manifold. Then there exists a real $\mathcal{P}$-oriented (resp., complex) $\mathcal{L}$-free G-vector bundle ν over F with $\mathcal{O}\ell(F, \nu) = 0$.*

PROOF. Let (U, V) be a special pair of real (resp., complex) G-modules U and V. By subtracting from U and V the trivial summands of dimension $\dim U^G$, we may assume that $\dim U^G = 0$ and $\dim V^G = 1$. Now, take a real (resp., complex) stable tangent bundle τ_F^{st} and a real (resp., complex) stable normal bundle ν_F^{st}. Then, consider the following real (resp., complex) G-vector bundles over F (cf. [36, the proof of Lemma 3.2 (a)]):

$$\eta = (\tau_F^{\mathrm{st}} \otimes \varepsilon_F^V) \oplus (\nu_F^{\mathrm{st}} \otimes \varepsilon_F^U) \quad \text{and} \quad \nu = (\tau_F^{\mathrm{st}} \otimes \varepsilon_F^{V-V^G}) \oplus (\nu_F^{\mathrm{st}} \otimes \varepsilon_F^U).$$

Clearly, $\eta \cong \tau_F^{\mathrm{st}} \oplus \nu$ as real (resp., complex) G-vector bundles. As U and V are $\mathcal{P}$-matched, $\mathcal{O}\ell(F, \nu) = \mathcal{O}\ell(\eta) = 0$ and thus we can apply Proposition 16 to ensure that ν is $\mathcal{P}$-oriented. In the case U and V are complex and the stable bundles τ_F^{st} and ν_F^{st} are complex, ν is a complex G-vector bundle by definition. Finally, note that ν is $\mathcal{L}$-free because so are U and $V - V^G$. □

Example 31 Let G be a finite nontrivial perfect group. Then $O^p(G) = G$ for each prime p, and thus $\mathcal{L}(G) = \{G\}$. As a result, for any G-module V, the nontrivial summand $V - V^G$ is $\mathcal{L}$-free. Assume that G has an element g of order pq for two distinct primes p and q. Set $n = pq$ and denote by ζ_n be the primitive n-th root of unity. Let $C = \langle\, g \,\rangle$, the cyclic subgroup of G generated by g. Set $U = U_1 \oplus U_2$ and $V = V_1 \oplus V_2$, where U_i and V_i $(i = 1, 2)$ are the irreducible 1-dimensional complex C-modules with characters

$$\begin{aligned} \chi_U(g) &= \chi_{U_1}(g) + \chi_{U_2}(g) = \zeta_n^p + \zeta_n^q \\ \chi_V(g) &= \chi_{V_1}(g) + \chi_{V_2}(g) = 1 + \zeta_n^{p+q}. \end{aligned}$$

Then the complex C-modules U and V are $\mathcal{P}$-matched. Moreover, $\dim U^C = 0$ and $\dim V^C = 1$. Hence, (U, V) is a special pair of complex C-modules and the induced pair $(\mathrm{Ind}_C^G(U), \mathrm{Ind}_C^G(V))$ is a special pair of complex G-modules. If G has a pq-dihedral subquotient, it follows from [36, the proof of Lemma 3.1 (b)] that there are two real $\mathcal{P}$-matched G-modules U and V with $\dim U^G = 0$ and $\dim V^G = 1$, proving that (U, V) is a special pair of real G-modules. □

For a finite group G with a quotient G/H isomorphic to $\mathbb{Z}_{pqr}$ or $\mathbb{Z}_{pqr} \times \mathbb{Z}_{pqr}$ for three distinct primes p, q, and r, we give examples of special pairs (U, V) of complex G-modules U and V with $\dim U = \dim V = 3$ or $\dim U = \dim V = 2$. In the case $G/H \cong \mathbb{Z}_{pqr}$, the example goes back to [31, Example 1.5].

Example 32 Let $n = pqr$ for three distinct primes p, q, and r. Let ζ_n be the primitive n-th root of unity. Let $G = \mathbb{Z}_n \times \mathbb{Z}_n = \langle\, a, b \,|\, a^n = 1,\ b^n = 1 \,\rangle$. Set $U = U_1 \oplus U_2$ and $V = V_1 \oplus V_2$, where U_i and V_i $(i = 1, 2)$ are the irreducible 1-dimensional complex G-modules with characters

$$\chi_U(g) = \chi_{U_1}(g) + \chi_{U_2}(g) = \begin{cases} \zeta_n^r + \zeta_n^{pq} & \text{when} \quad g = a \\ 1 + \zeta_n & \text{when} \quad g = b \end{cases}$$

$$\chi_V(g) = \chi_{V_1}(g) + \chi_{V_2}(g) = \begin{cases} 1 + \zeta_n^{pq+r} & \text{when} \quad g = a \\ 1 + \zeta_n & \text{when} \quad g = b \end{cases}$$

Now, let $G = \mathbb{Z}_n = \langle\, a \,|\, a^n = 1 \,\rangle$. Set $U = U_1 \oplus U_2 \oplus U_3$ and $V = V_1 \oplus V_2 \oplus V_3$, where (as in [31]) U_i and V_i $(i = 1, 2, 3)$ are the irreducible 1-dimensional complex G-modules with characters

$$\begin{aligned} \chi_U(a) &= \chi_{U_1}(a) + \chi_{U_2}(a) + \chi_{U_3}(a) = \zeta_n^x + \zeta_n^y + \zeta_n^z \\ \chi_V(a) &= \chi_{V_1}(a) + \chi_{V_2}(a) + \chi_{V_3}(a) = 1 + \zeta_n + \zeta_n \end{aligned}$$

and the integers x, y, and z are chosen so that the following holds:

$$\begin{array}{lll} x \equiv 0 \pmod p, & x \equiv 1 \pmod q, & x \equiv 1 \pmod r \\ y \equiv 1 \pmod p, & y \equiv 0 \pmod q, & y \equiv 1 \pmod r \\ z \equiv 1 \pmod p, & z \equiv 1 \pmod q, & z \equiv 0 \pmod r. \end{array}$$

In both cases, a straightforward verification shows that the complex G-modules U and V are $\mathcal{P}$-matched and for each $H \in \mathcal{L}(G)$, $\dim U^H = 0$ and $\dim V^H = 1$. Therefore, (U, V) is a special pair of complex G-modules. Let G be a finite group with a quotient G/H isomorphic to $\mathbb{Z}_n$ or $\mathbb{Z}_n \times \mathbb{Z}_n$. Then, by making use of the epimorphism $G \to G/H$, the pair (U, V) of G/H-modules U and V constructed above becomes a special pair of complex G-modules. □

Theorem 33 *Let G be a finite perfect group with a pq-element, or a finite Oliver group with a pqr-cyclic quotient. Let F be a closed smooth manifold. Assume that F is stably complex and each connected component of F is simply connected or stably parallelizable. Then F is the fixed point set of a smooth $\mathcal{P}$-typical action of G on some sphere S. Moreover, $\nu_{F \subset S}$ can be chosen to be the realification of a complex $\mathcal{L}$-free G-vector bundle.*

PROOF. If G is a finite perfect group with a pq-element, G has a special pair of complex G-modules by Example 31. In turn, if G is a finite Oliver group with a pqr-cyclic quotient, G has a special pair of complex G-modules by Example 32. Therefore, in both cases, Lemma 30 asserts that there exists a complex $\mathcal{L}$-free G-vector bundle ν over F with $\mathcal{O}\ell(F,\nu) = 0$.

If G is perfect, $O^2(G) = G$. If G has a pqr-cyclic quotient, $O^p(G) \neq G$, $O^q(G) \neq G$, and $O^r(G) \neq G$, where say p and q are odd primes. As a result, in both cases, G is a gap group by [32]. As $\mathcal{O}\ell(F,\nu) = 0$ and ν is complex, the result follows by Corollary 29. □

For any integer $m \geq 0$, let Ω^m_{SO} (resp. Ω^m_U) be the group of all cobordism classes of oriented (resp. stably complex) closed smooth manifolds of dimension m.

Theorem 34 *Let G be a finite perfect group with a pq-element, or let G be a finite Oliver group with a pqr-cyclic quotient. Then, for any integer $m \geq 0$, each element in $\Omega^m_{SO}/\mathrm{tor}$ is represented by a stably complex smooth manifold F which is the fixed point set of a smooth action of G on some sphere.*

PROOF. By [48, p. 130], each class of Ω^m_{SO} is represented by an oriented closed smooth manifold whose connected components are all simply connected. By [48, p. 180], the composition $\Omega^m_U \to \Omega^m_{SO} \to \Omega^m_{SO}/\mathrm{tor}$ of the forgetful map and the quotient map is an epimorphism. Thus, in $\Omega^m_{SO}/\mathrm{tor}$, each element is represented by a stably complex closed smooth manifold F whose connected components are all simply connected. According to Theorem 33, F is the fixed point set of a smooth action of G on some sphere. □

8 Proofs of Theorems 1–6 stated in the introduction

Proof of Theorem 1. If a finite group G acts smoothly on a sphere S, then the fixed point set $F = S^G$ is a closed smooth manifold. Moreover, if the action of G on S is $\mathcal{P}$-typical with G not of prime power order, then by Lemma 7, $\mathcal{O}\ell(F,\nu) = 0$ for $\nu = \nu_{F \subset S}$. So, in Theorem 1, (1) implies (2). The converse implication, (2) implies (1), follows from Theorem 27. □

Proof of Theorem 2. Theorem 2 follows immediately from Theorem 1. □

Proof of Theorem 3. If a finite group G (not of prime power order) acts smoothly on a sphere S so that the G-fixed point set S^G contains at least two points and the action of G on S is $\mathcal{P}$-typical, then by Smith theory, the P-fixed point set S^P is connected for each $P \in \mathcal{P}(G)$. In particular, for any two points $x, y \in S^G \subset S^P$, the P-modules $\mathrm{Res}^G_P(T_x(S))$ and $\mathrm{Res}^G_P(T_y(S))$ are isomorphic. Hence, in Theorem 3, (1) implies (2). The converse implication, (2) implies (1), follows from Theorem 28. □

Proof of Theorem 4. Let G be a finite nontrivial perfect group. Let F be a closed smooth manifold. If F is the fixed point set of a smooth $\mathcal{P}$-typical action of G on a sphere S, then by Lemma 7, $\mathcal{O}\ell(F,\nu) = 0$ for $\nu = \nu_{F\subset S}$. Thus, by Theorem 12, F is the fixed point set of a smooth action of G on a disk D, proving that in Theorem 4, (1) implies (2). In order to prove that (2) implies (1), assume that F is the fixed point set of a smooth action of G on a disk D. According to [36], $\mathcal{O}\ell(F,\nu) = 0$ for $\nu = \nu_{F\subset D}$ (cf. the proof of Lemma 7). As $O^p(G) = G$ for each prime p, $\mathcal{L}(G) = \{G\}$ and thus ν is $\mathcal{L}$-free. So, if each connected component of F is simply connected or stably parallelizable, Corollary 29 asserts that there exists a smooth $\mathcal{P}$-typical action of G on a sphere S such that $S^G = F$, proving that (2) implies (1). Clearly, (2) and (3) are equivalent by Theorem 12. □

Proof of Theorem 5. In Theorem 5, (1) implies (3) by Proposition 10, and (3) implies (1) by Theorem 33. Moreover, (2) and (3) are equivalent by Theorem 12. □

Proof of Theorem 6. Let G be a finite perfect group with a pq-element, or let G be a finite Oliver group with a pqr-cyclic quotient. For an oriented closed smooth manifold M of dimension $2k$ (resp., $4k$) with $k \geq 0$, consider the element $[M]$ in $\Omega_{SO}^{2k}/\mathrm{tor}$ (resp., $\Omega_{SO}^{4k}/\mathrm{tor}$) determined by M. According to Theorem 34, $[M]$ is represented by a stably complex (and thus oriented) closed smooth manifold F of dimension $2k$ (resp., $4k$) which is the fixed point set of a smooth action of G on some sphere. As $[M] = [F]$, M and F have the same Chern (resp., Pontrjagin) numbers. □

Appendix: Generalization of Deleting–Inserting Theorem Added by Masaharu Morimoto

For a finite group G, in accordance with G-module $\mathcal{P}$-orientability, a smooth G-manifold X is called *$\mathcal{P}$-oriented* if for each $P \in \mathcal{P}(G)$, each connected component of X^P is oriented and the transformation

$$g : X^P \to X^{gPg^{-1}},\ x \mapsto gx$$

is orientation preserving for any $g \in G$, or equivalently, the transformation $g : X^P \to X^P$, $x \mapsto gx$ is orientation preserving for any $g \in N_G(P)$.

In [28], for a finite Oliver group G, we obtained a deleting–inserting theorem about the fixed point sets of smooth actions of G on spheres. We considered a smooth action of G on a homotopy sphere Y under the condition that at some $y_0 \in Y^G$, the tangent G-module $T_{y_0}(Y)$ is G-oriented. In this appendix, we prove that a similar deleting–inserting theorem holds under the weaker condition that $T_{y_0}(Y)$ is $\mathcal{P}$-oriented. We recall that for a subgroup H of G, $Y^{=H}$ consisits of all points $y \in Y$ with isotropy subgroup $G_y = H$. In general, $Y^{=H}$ may have connected components of different dimensions, and in such a case, we define $\dim Y^{=H}$ as the maximum of the dimensions of all connected components of $Y^{=H}$. Similarly, $\dim Y^H$ is the maximum of the dimensions of all connected components of Y^H. The deleting–inserting theorem holds under a weak gap condition imposed on Y that reads as follows.

Following [28], we say that Y satisfies the *weak gap condition* (WGC) if

$$d_Y(P,H) = \dim Y^P - 2\dim Y^H \geq 0$$

for all subgroups $P < H \leq G$ with $P \in \mathcal{P}(G)$, and the following claims hold.

(1) If $d_Y(P,H) = 0$, then $|H : P| = 2$ and Y^H is connected, oriented, and the transformation $g : Y^H \to Y^H$ is orientation preserving for any element $g \in N_G(H)$, and $\dim Y^H > \dim Y^K + 1$ whenever $H < K \leq G$.

(2) If $d_Y(P,H) = d_Y(P,H') = 0$, then the subgroup $\langle H, H' \rangle$ of G generated by H and H' is not a large subgroup of G (i.e., $\langle H, H' \rangle \notin \mathcal{L}(G)$).

Remark 35 Note that Y satisfies the weak gap condition (WGC) provided Y satisfies the gap condition (GC) asserting that

$$d_Y(P,H) = \dim Y^P - 2\dim Y^H > 0$$

for all subgroups $P < H \leq G$ with $P \in \mathcal{P}(G)$.

Now, we wish to present the announced generalization of [28, Theorem 0.1]. In Theorem 18, under the gap condition (GC) imposed on Y, we restate the deleting part of this deleting–inserting theorem.

Theorem 36 (cf. [28, Theorem 0.1]) *Let G be a finite Oliver group. Let Y be a closed smooth G-manifold whose underlying manifold is a homotopy sphere of dimension $n \geq 5$. Let $F_1, \ldots, F_t$ be the connected components of Y^G and let $n_1, \ldots, n_t$ be integers ≥ 0. Suppose the following five conditions (1)–(5):*

(1) Y satisfies the weak gap condition (WGC).

(2) $\dim Y^P \geq 5$ and $\dim Y^{=H} \geq 2$ for any $P \in \mathcal{P}(G)$ and $H \in \mathcal{PC}(G)$.

(3) Y^P is simply connected for any $P \in \mathcal{P}(G)$.

(4) The tangent G-module $T_{y_0}(Y)$ is $\mathcal{P}$-oriented for some $y_0 \in Y^G$.

(5) $n_i = n_j$ when some connected component of Y^H contains both F_i and F_j for some $H \in \mathcal{L}(G)$, where $1 \leq i, j \leq t$.

Then there exists a smooth action of G on the sphere $S = S^n$ such that S^G has the form of the disjoint union of copies of F_i's:

$$S^G = \coprod_{i=1}^{t} \coprod_{j=1}^{n_i} F_{i,j} \quad \textit{and} \quad \nu_{F_{i,j} \subset S} \cong \nu_{F_i \subset Y} \quad \textit{as } G\textit{-vector bundles,}$$

where $F_{i,j} = F_i$ for $j = 1, \ldots, n_i$ and $i = 1, \ldots, t$. In particular, $n_i = 0$ means that F_i is deleted from Y^G, and $n_i = 1$ means that F_i is preserved, while $n_i > 1$ means that $(n_i - 1)$ of new copies of F_i are inserted.

PROOF. The proof of Theorem 36 is essentially the same as the proof of [28, Theorem 0.1]. Thus, we describe only the parts where formal differences in the proofs arise. In [28, Theorem 0.1], the G-orientation of $T_{y_0}(Y)$ is used in the following four procedures:

A: To guarantee Lemma 3.2 (3.2.3) in [28].

B: Definition of G-connected sum (cf. (1.1.1) and (1.1.2) in [28]).

C: Determining the surgery obstruction groups [28, Cases 2 and 3, pp. 26–29]

$$\mathcal{O}(N_G(H)/H, Y^H, \mathbb{Z}_{(p)}) \quad \text{and} \quad \mathcal{O}(G, Y, \mathbb{Z}).$$

D: To guarantee the equalities [28, Cases 2 and 3, pp. 26–29]

$$\sigma(((1-\kappa)f)^H, ((1-\kappa)b)^H) = \mathrm{Fix}_H(1-\kappa)\sigma(f^H, b^H), \quad \text{and}$$

$$\sigma((1-\kappa)f, (1-\kappa)b) = (1-\kappa)\sigma(f, b).$$

Therefore, in order to complete the proof, we should show that the replacement of G-orientability by $\mathcal{P}$-orientability does not affect Procedures A–D.

Procedure A. In the proof of [28, Lemma 3.2 (3.2.3)], the relevant group action was the H/P-action on Y^P for $P \in \mathcal{P}(G)$ and $P \trianglelefteq H$ with H/P cyclic. Thus, for the conclusion, G-orientablity of $T_{y_0}(Y)$ is superfluous and $\mathcal{P}$-orientablity of $T_{y_0}(Y)$ is sufficient.

Procedure B. The G-connected sums $X\#(G \times_H X)$ and $X\#(G \times_H (-X))$ in [28] were performed not with arbitrary H-diffeomorphisms ϕ but with the specific H-diffeomorphisms

$$f|_{U_{a(H,-)}}^{-1} \circ f|_{U_{a(H,+)}} : U_{a(H,+)} \to U_{a(H,-)}$$

and

$$f|_{U_{a(H,+)}}^{-1} \circ f|_{U_{a(H,+)}} : U_{a(H,+)} \to U_{a(H,+)},$$

where $a(H,-)$ and $a(H,+)$ are the points appearing in [28, Lemma 2.3 (2.3.7)], U_{y_0}, $U_{a(H,-)}$, $U_{a(H,+)}$ are H-slice neighborhoods of y_0, $a(H,-)$, $a(H,+)$ in X, and $f|_{U_{a(H,-)}} : U_{a(H,-)} \to U_{y_0}$ and $f|_{U_{a(H,+)}} : U_{a(H,+)} \to U_{y_0}$ are restrictions of the G-map f in [28, Lemma 2.3]. As a result, we can perform the required G-connected sum for f and also for G-maps obtained from it by G-connected sum and G-surgery even when the G-orientability of $T_{y_0}(Y)$ is not supplied. Hence, without G-orientablity of $T_{y_0}(Y)$, we can perform the G-connected sum $(1+(-\beta)^{\%})X$ to modify $\chi(X^L)$ in [28, Case 1, pp. 25–26] and to modify $[H_k(X;\mathbb{Z})]$ in $\widetilde{K}_0(\mathbb{Z}[G])$ in [28, Case 3, p. 28].

Procedure C. In Cases 2 and 3 of the proof of [28, Theorem 0.1], H is in $\mathcal{P}(G)$. Thus, if $T_{y_0}(Y)$ is $\mathcal{P}$-orientable, then X^H in [28, Lemma 2.3] is orientable so that $g : X^H \to X^{gHg^{-1}}$ is orientation preserving for any $g \in G$. Consequently, each G-manifold X appearring in the proof of [28, Theorem 0.1] has the same property and the relevant orientation homomorphism $w : N_G(H)/H \to \{\pm 1\}$ given by the $N_G(H)/H$-action on X^H is trivial when $T_{y_0}(Y)$ is $\mathcal{P}$-orientable. So, the obstruction groups under the $\mathcal{P}$-orientability of $T_{y_0}(Y)$ are the same as those under the G-orientability of $T_{y_0}(Y)$.

Procedure D. Here, $H \in \mathcal{P}(G)$. If we assume the $\mathcal{P}$-orientability at y_0, then by the same argument as in Procedure C, we can obtain the same formulae of the surgery obstructions as in [28, pp. 27 and 29], i.e., we obtain the equality $\sigma(((1-\kappa)f)^H, ((1-\kappa)b)^H) = \mathrm{Fix}_H(1-\kappa)\sigma(f^H, b^H)$, as well as the equality $\sigma((1-\kappa)f, (1-\kappa)b) = (1-\kappa)\sigma(f,b)$. □

Note that if $\nu_{F_i \subset Y}$ and $\nu_{F_j \subset Y}$ are $\mathcal{L}$-free, F_i and F_j are not contained in the same connected component of Y^H for any $H \in \mathcal{L}(G)$. Thus the restriction in (5) does not occur when $\nu_{F \subset Y}$ is $\mathcal{L}$-free.

References

[1] C. Allday and V. Puppe, Cohomological methods in transformation groups, Cambridge Univ. Press, Cambridge, 1993.

[2] A. Assadi, Finite group actions on simply-connected manifolds and CW complexes, Mem. Amer. Math. Soc. 35, No. 257, Amer. Math. Soc., Providence, 1982.

[3] A. Assadi and W. Browder, On the existence and classification of extensions of actions on submanifolds of disks and spheres, Trans. Amer. Math. Soc. 291 (1985) 487–502.

[4] A. Bak and M. Morimoto, K-theoretic groups with positioning map and equivariant surgery, Proc. Japan Acad. 70 (1994) 6–11.

[5] A. Bak and M. Morimoto, Equivariant surgery with middle dimensional singular sets. I, Forum Math. 8 (1996) 267–302.

[6] G. E. Bredon, Introduction to compact transformation groups, Pure and Applied Mathematics, Vol. 46, Academic Press, New York, 1972.

[7] W. Browder, Surgery and the theory of differentiable transformation groups, in: Proc. Conference on Transformation Groups, New Orleans, 1967, P. S. Mostert (Ed.), Springer-Verlag, Berlin, 1968, pp. 1–46.

[8] W. Browder and F. Quinn, A surgery theory for G-manifolds and stratified sets, in: Manifolds – Tokyo 1973, Proceedings of the International Conference, Tokyo, 1973, A. Hattori (Ed.), Univ. of Tokyo Press, Tokyo, 1975, pp. 27–36.

[9] S. E. Cappell and S. Weinberger, Surgery theoretic methods in group actions, in: Surveys on Surgery Theory, Vol. 2, S. Cappell, A. Ranicki and J. Rosenberg (Eds.), Ann. Math. Studies, No. 149, Princeton Univ. Press, Princeton, 2001, pp. 285–317.

[10] J. F. Davis and T. tom Dieck, Some exotic dihedral actions on spheres, Indiana Univ. Math. J. 37 (1988) 431–450.

[11] J. F. Davis and S. Weinberger, Group actions on homology spheres, Invent. Math. 86 (1986) 209–231.

[12] T. tom Dieck, Transformation Groups, de Gruyter Studies in Mathematics 8, Walter de Gruyter, Berlin, 1987.

[13] K. H. Dovermann, Diffeomorphism classification of finite group actions on disks, Topology and its Appl. 16 (1983) 123–133.

[14] K. H. Dovermann and M. Herzog, Gap conditions for representations of symmetric groups, J. Pure and Appl. Algebra 119 (1997) 113–137.

[15] K. H. Dovermann and T. Petrie, G-surgery II, Mem. Amer. Math. Soc. 37, No. 260, Amer. Math. Soc., Providence, 1982.

[16] K. H. Dovermann and T. Petrie, An induction theorem for equivariant surgery, G-surgery III, Amer. J. Math. 105 (1983) 1369–1403.

[17] K. H. Dovermann, T. Petrie, and R. Schultz, Transformation groups and fixed point data, in: Group Actions on Manifolds, R. Schultz (Ed.), Contemp. Math. 36, Amer. Math. Soc., Providence, 1985, pp. 159–189.

[18] A. L. Edmonds and R. Lee, Fixed point sets of group actions on Euclidean space, Topology 14 (1975) 339–345.

[19] A. L. Edmonds and R. Lee, Compact Lie groups which act on Euclidean space without fixed points, Proc. Amer. Math. Soc. 55 (1976) 416–418.

[20] L. E. Jones, The converse to the fixed point theorem of P. A. Smith: I, Ann. Math. 94 (1971) 52–68.

[21] K. Kawakubo, The Theory of Transformation Groups, Oxford University Press, Oxford, 1991.

[22] M. A. Kervaire and J. W. Milnor, Groups of homotopy spheres I, Ann. Math. 77 (1963) 504–537.

[23] A. A. Kosinski, Differential manifolds, Pure and Applied Mathematics, Vol. 138, Academic Press, New York, 1993.

[24] E. Laitinen, M. Morimoto and K. Pawałowski, Deleting–Inserting Theorem for smooth actions of finite nonsolvable groups on spheres, Comment. Math. Helv. 70 (1995) 10–38.

[25] E. Laitinen and M. Morimoto, Finite groups with smooth one fixed point actions on spheres, Forum Math. 10 (1998) 479–520.

[26] M. Morimoto, Positioning map, equivariant surgery obstruction, and applications, in: Transformation Groups and Algebraic Topology, Y. Kitada (Ed.), Kyoto Univ., RIMS Kokyuroku 793, 1992, pp. 75–93.

[27] M. Morimoto, Equivariant surgery theory: Construction of equivariant normal maps, Publ. Res. Inst. Math. Sci. 31 (1995) 145–167.

[28] M. Morimoto, Equivariant surgery theory: Deleting–Inserting Theorem of fixed point manifolds on spheres and disks, K-Theory 15 (1998) 13–32.

[29] M. Morimoto, Equivariant surgery with middle dimensional singular sets. II: Equivariant framed cobordism invariance, Trans. Amer. Math. Soc. 353 (2001) 2427–2440.

[30] M. Morimoto and K. Pawałowski, Equivariant wedge sum construction of finite contractible G-CW complexes with G-vector bundles, Osaka J. Math. 36 (1999) 767–781.

[31] M. Morimoto and K. Pawałowski, The Equivariant Bundle Subtraction Theorem and its applications, Fund. Math. 161 (1999) 279–303.

[32] M. Morimoto, T. Sumi and M. Yanagihara, Finite groups possessing gap modules, in: Geometry and Topology: Aarhus, K. Grove, I.H. Madsen and E.K. Pedersen (Eds.), Contemp. Math. 258, Amer. Math. Soc., Providence, 2000, pp. 329–342.

[33] R. Oliver, Fixed point sets of group actions on finite acyclic complexes, Comment. Math. Helv. 50 (1975) 155–177.

[34] R. Oliver, Smooth compact Lie group actions on disks, Math. Z. 149 (1976) 79–96.

[35] R. Oliver, G-actions on disks and permutation representations II, Math. Z. 157 (1977) 237–263.

[36] B. Oliver, Fixed point sets and tangent bundles of actions on disks and Euclidean spaces, Topology 35 (1996) 583–615.

[37] K. Pawałowski, Fixed point sets of smooth group actions on disks and Euclidean spaces. A survey, in: Geometric and Algebraic Topology, H. Toruńczyk, S. Jackowski and S. Spież (Eds.), Banach Center Publications, Vol. 18, PWN–Polish Scientific Publishers, Warsaw, 1986, pp. 165–180.

[38] K. Pawałowski, Equivariant thickening for compact Lie group actions, Mathematica Gottingensis, Heft 71, Göttingen, 1986.

[39] K. Pawałowski, Fixed point sets of smooth group actions on disks and Euclidean spaces, Topology 28 (1989) 273–289; Corrections: ibid. 35 (1996) 749–750.

[40] K. Pawałowski, Chern and Pontrjagin numbers in perfect symmetries of spheres, K-Theory 13 (1998) 41–55.

[41] K. Pawałowski and R. Solomon, Smith equivalence and finite Oliver groups with Laitinen number 0 or 1, to appear in Algebraic and Geometric Topology (2002).

[42] T. Petrie, Pseudoequivalences of G-manifolds, in: Algebraic and Geometric Topology, R. J. Milgram (Ed.), Proc. Sympos. Pure Math., Vol. XXXII – Part 1, Amer. Math. Soc., Providence, 1978, pp. 169–210.

[43] T. Petrie, G-surgery I – A survey, in: Algebraic and Geometric Topology, Proceedings, Santa Barbara 1977, K. C. Millett (Ed.), Lecture Notes in Math. 664, Springer-Verlag, Berlin, 1978, pp. 197–242.

[44] T. Petrie, One fixed point actions on spheres I, Adv. Math. 46 (1982) 3–14; II, ibid. 15–70.

[45] T. Petrie and J. D. Randall, Transformation Groups on Manifolds, Marcel Dekker, New York, 1984.

[46] M. Rothenberg and J. Sondow, Nonlinear smooth representations of compact Lie groups, Pac. J. Math. 84 (1979) 427–444.

[47] R. Schultz, Pontrjagin numbers and periodic diffeomorphisms of spheres, in: Transformation Groups, Proceedings, Osaka 1987, K. Kawakubo (Ed.), Lecture Notes in Math. 1375, Springer-Verlag, Berlin, 1989, pp. 307–318.

[48] R. E. Stong, Notes on cobordism theory, Mathematical Notes, Princeton University Press and the University of Tokyo Press, Princeton and Tokyo, 1968.

[49] T. Sumi, Gap modules for direct product groups, J. Math. Soc. Japan 53 (2001) 975–990.

[50] S. Weinberger, Constructions of group actions: A survey of some recent developments, in: Group Actions on Manifolds, R. Schultz (Ed.), Contemp. Math. 36, Amer. Math. Soc., Providence, 1985, pp. 269–298.