MANIFOLDS AS THE FIXED POINT SETS OF SMOOTH COMPACT LIE GROUP ACTIONS

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ABSTRACT. This article describes progress made in answering the following question. Given a compact Lie group G, which manifolds F occur as the fixed point sets of smooth actions of G on specific manifolds E such as Euclidean spaces (resp., disks, spheres)? For actions on Euclidean spaces and disks, the answer is obtained when G is a finite group, a torus, or an extension of a finite p-group by a torus. For actions on spheres, the answer is much less complete and the cases where it is known are discussed. Moreover, equivariant techniques for constructing smooth actions of G on Euclidean spaces, disks, and spheres are described such as thickening, surgery, and vector bundle extension and subtraction.

1. PROLOGUE

The theory of transformation groups includes as a mainstream topic the study of smooth G-manifolds for compact Lie groups G. By definition, a *smooth G-manifold* is a smooth manifold M with a *smooth action of* G, i.e., a smooth map $G \times M \to M, (g, x) \mapsto gx$ such that $ex = x$ and $(gh)x = g(hx)$ for the neutral element $e \in G$ and all $g, h \in G$, $x \in M$. One of the basic problems is to describe the set of points in M left fixed by the action of G,

$$F = M^G = \{x \in M : \ gx = x \text{ for all } g \in G\}.$$

It turns out that either $F = \varnothing$ or F is a smooth submanifold of M such that $\partial F = F \cap \partial M$ (see the Slice Theorem [34, Thm. (5.6) on p. 40], [61, Thm. 4.10 on p. 184], and [15, p. 171 and Cor. 2.4 on p. 308]). In particular, $\partial F = \varnothing$ when $\partial M = \varnothing$. Moreover, F is compact when M is compact, and thus F is closed when M is closed. In the case $F \neq \varnothing$, another basic problem is to describe the equivariant normal bundle of F in M, which gives a complete information about the behaviour of the action of G on an invariant open neighborhhod of F in M (see the Equivariant Tubular Neighborhood Theorem [15, Thm. 2.2 on p. 306] and [61, Thm. 4.8 on p. 178] for the precise statements of the corresponding results).

Our focus is on the description of the fixed point set. We consider the following question. *What are the necessary and sufficient conditions for a smooth manifold F to occur as the fixed point set of a smooth action of a given compact Lie group G on a specific manifold?* The case $F = \varnothing$ leads to the next question. *Which compact Lie groups G have smooth fixed point free actions on specific manifolds?* A lot of the effort has been made toward trying to answer both questions for actions on Euclidean spaces, disks, and spheres. In this article, we survey the related results obtained for actions on these manifolds.

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First, note that any compact Lie group G has a linear fixed point free action on a sphere. To obtain such action, take a *real G-module* V, which by definition is a finite dimensional real vector space with a *linear action* of G, i.e., a continuous map $G \times V \to V$, $(g, x) \mapsto gx$ such that $ex = x$, $(gh)x = g(hx)$, and the transformations $V \to V$, $x \mapsto gx$ are linear for all $g \in G$. After choosing a G-invariant inner product on V, we may assume that these transformations are orthogonal (cf. [15, Thm. 3.5 on p. 14]). By letting $D(V)$ and $S(V)$ denote, respectively, the unit disk and sphere in V with respect to the chosen G-invariant inner product on V, the orthogonal action of G on V restricts to orthogonal actions of G on $D(V)$ and $S(V)$. Once we choose V so that the fixed point set $V^G = \{0\}$, the resulting action of G on $S(V)$ is fixed point free (cf. [15, Prop. 3.6 on p. 15]).

We recall that for a compact Lie group G, any real G-module V is a smooth G-manifold. In fact, as the transformations $V \to V$, $x \mapsto gx$ are linear for all $g \in G$, they are smooth, and thus the action $G \times V \to V$, $(g, x) \mapsto gx$ is smooth (see [77] and [15, p. 298]).

We also recall that an action of G on a manifold M is called *effective* if for any $g \in G$ with $g \neq e$, the transformation $M \to M$, $x \mapsto gx$ differs from the identity on M. Moreover, a real G-module V is called *faithful* if the action of G on V is effective.

When constructing smooth actions of G on Euclidean spaces or disks with prescribed fixed point sets, one may often reduce to the case where the group G is much less complicated. This can be done using the following arguments.

Let F be a smooth manifold or $F = \varnothing$. For a normal subgroup $H \trianglelefteq G$, assume that the quotient group G/H has a smooth action on a smooth manifold M such that the fixed point set $M^{G/H} = F$. Then, via the quotient map $G \to G/H$, we get a smooth action of G on M with fixed point set $M^G = F$. Of course, the resulting action may not be effective. However, we can take a real faithful G-module V with $V^G = \{0\}$ and consider the cartesian product $M \times V$ with the diagonal action of G. Then the resulting action is effective and $(M \times V)^G = F \times \{0\} = F$. Similarly, after choosing a G-invariant inner product on V, we obtain a smooth effective action of G on $M \times D(V)$ such that $(M \times D(V))^G = F \times \{0\} = F$. As a result, the following proposition holds.

Proposition 1.1. *Let G be a compact Lie group. If some quotient of G acts smoothly on a Euclidean space (resp., disk) with fixed point set F, then G has a smooth effective action on a Euclidean space (resp., disk) with the same fixed point set F.*

Let G be a compact Lie group. For any subgroup $H \leq G$ of finite index $|G : H|$, we can obtain smooth G-manifolds from smooth H-manifolds by applying a topological analogue of the induction procedure in representation theory. Briefly, we can proceed as follows.

For a smooth H-manifold E with $E^H = F$, consider the space $\mathrm{Ind}_H^G(E)$ of maps $\varphi : G \to E$ with $\varphi(hx) = h\varphi(x)$ for all $h \in H$, $x \in G$. Equip $\mathrm{Ind}_H^G(E)$ with the compact-open topology and define an action of G on $\mathrm{Ind}_H^G(E)$ by $(g\varphi)(x) = \varphi(xg)$ for all $g, x \in G$. Then $\mathrm{Ind}_H^G(E)$ is homeomorphic to the cartesian product $E^{\times n}$ of $n = |G : H|$ copies of E, and the action of G on $\mathrm{Ind}_H^G(E)$ yields a smooth action of G on $E^{\times n}$ such that $(E^{\times n})^G$ is the image of F under the diagonal map $E \to E^{\times n}$. Therefore $(E^{\times n})^G$ is diffeomorphic to F. Clearly, if E is a Euclidean space (resp., disk), so is $E^{\times n}$. This proves the next proposition.

Proposition 1.2. *Let G be a compact Lie group. If some subgroup $H \leq G$ of finite index acts smoothly on a Euclidean space (resp., disk) with fixed point set F, then G has a smooth action on a Euclidean space (resp., disk) with the same fixed point set F.*

Propositions 1.1 and 1.2 both are true for actions on Euclidean spaces and disks. While trying to prove similar statements for actions on spheres, we cannot copy the arguments because a cartesian product of spheres is not a sphere. At the moment, the statements for actions on spheres are not verified and are considered as challenging open problems.

Problem 1.3. *Let G be a compact Lie group. Assume that G has a quotient acting smoothly on a sphere with fixed point set F. Does G have a smooth effective action on a sphere with fixed point set diffeomorphic to F?*

Problem 1.4. *Let G be a compact Lie group. Assume that G has a finite index subgroup acting smoothly on a sphere with fixed point set F. Does G have a smooth action on a sphere with fixed point set diffeomorphic to F?*

We recall that for a compact Lie group G, any smooth G-manifold M admits the structure of a G-CW complex (i.e., a G-space built up from G-cells). Moreover, the complex is finite (i.e., it has finitely many G-cells) if and only if M is compact (see [53] and [72]).

2. Fixed point free actions on Euclidean spaces and disks

The question whether contractible manifolds (in particular, Euclidean spaces and disks) admit smooth fixed point free actions of compact Lie groups goes back to P.A. Smith [116]. As we see in Theorems 2.1 and 2.2 below, this question has been answered completely.

Let G be a compact Lie group and let G_0 be the connected component of G containing the neutral element of G. Then G_0 is a normal subgroup of G of finite index. Consider the related exact sequence $1 \to G_0 \to G \to G/G_0 \to 1$. We say that G is *from Smith theory* if G_0 is an abelian group (i.e., G_0 is the trivial group or a torus) and G/G_0 is of prime power order (i.e., $G = G_0$ or G/G_0 is a p-group for some prime p). Thus, a finite group G is not from Smith theory if and only if G is not of prime power order.

For any contractible smooth G-manifold M, the fixed point set M^G is connected and nonempty when G is from Smith theory (cf. Section 7). Hence, if G is from Smith theory, G has no smooth fixed point free action on a contractible manifold.

By generalizing the construction of Conner and Montgomery [26] performed for the group $G = SO(3)$, Hsiang and Hsiang [50] have shown that any nonabelian compact connected Lie group G has a smooth fixed point free action on a Euclidean space. The construction of Conner and Floyd [24] modified by Kister [63] shows that if $G = \mathbb{Z}_{pq}$, the cyclic group of order pq for two relatively prime integers $p, q \geq 2$, then G has a smooth fixed point free action on a Euclidean space (see [15, pp. 58–61] for a description of the Conner–Floyd example). Edmonds and Lee [44] have proved that if G is a finite group with a surjection $G \to \mathbb{Z}_p$ and an injection $\mathbb{Z}_q \to G$ for two distinct primes p and q, then G has a smooth fixed point free action on a Euclidean space. Propositions 1.1 and 1.2 and the results described above yield the following theorem.

Theorem 2.1. ([24], [63], [44], [26], [50]) *A compact Lie group G has a smooth fixed point free action on some Euclidean space if and only if G is not from Smith theory.*

A smooth fixed point free action of a compact Lie group G on a disk yields similar action of G on Euclidean space. In fact, the action on the disk can be restricted to the interior of the disk or extended radially to an open collar added to the boundary of the disk, producing a smooth fixed point free action of G on Euclidean space of dimension of the disk.

Floyd and Richardson [46] constructed for the first time a smooth fixed point free action on a disk for $G = A_5$, the alternating group on five letters (see [15, pp. 55–58] for a description of the Floyd–Richardson example). In turn, Greever [47] has detected many finite solvable groups G not of prime power order (and thus not from Smith theory) which have no fixed point free actions on disks. Oliver [91], [92] has answered completely the question which compact Lie groups G have smooth fixed point free actions on disks. Before we recall the result, we adopt the following definition from [104]. For a finite group G, a series $P \trianglelefteq H \trianglelefteq G$ of normal subgroups P of H and H of G is called an *isthmus series* of subgroups of G if the groups P and G/H are of prime power order and the group H/P is cyclic.

Theorem 2.2. ([91], [92]) *A compact Lie group G has a smooth fixed point free action on a disk if and only if G_0 is nonabelian or there is no isthmus series of subgroups of G/G_0.*

A finite group G is called an *Oliver group* if G has no isthmus series of subgroups (cf. [65]). Any finite nonsolvable group G is an Oliver group. Moreover, a finite nilpotent group G is an Oliver group if and only if G has three or more noncyclic Sylow subgroups. For example, for any three distinct primes p, q, and r, the abelian group $G = \mathbb{Z}_{pqr} \times \mathbb{Z}_{pqr}$ is an Oliver group. The smallest Oliver group is A_5 of order 60, and the smallest solvable Oliver groups are $A_4 \times S_3$ and $S_4 \times \mathbb{Z}_3$ both of order 72 (here, A_4 is the alternating group on 4 letters, while S_3 and S_4 are the symmetric groups on 3 and 4 letters, respectively).

A compact Lie group G is called an *Oliver group* if G_0 is nonabelian or there is no isthmus series of subgroups of G/G_0. By Theorem 2.2, a compact Lie group G has a smooth fixed point free action on a disk if and only if G is an Oliver group.

3. Smooth one fixed point actions on spheres

In 1946, Montgomery and Samelson [76] made a comment that when a compact Lie group G acts smoothly on a sphere S^n in such a way as to have one fixed point, it is likely that there must be a second fixed point. In 1977, for $G = SL(2,5)$ or $SL(2,5) \times \mathbb{Z}_r$ with $(120, r) = 1$, Stein [120] constructed (using surgery methods) smooth actions of G on S^7 with exactly one fixed point, showing for the first time that the second fixed point may not occur.

Petrie [107] constructed smooth actions of G on homotopy spheres with exactly one fixed point for any finite abelian group G of odd order, with three or more noncyclic Sylow subgroups, and for $G = S^3$ or $SO(3)$. He announced the existence of similar actions also for the nonsolvable groups $G = SL(2,q)$ or $PSL(2,q)$, where $q \geq 5$ is a power of an odd prime. In [64], the last result is generalized to the case where G is a finite nonsolvable group.

Theorem 3.1. ([64]) *Any finite nonsolvable group G has a smooth action on a sphere with exactly one fixed point.*

Given a smooth action of a compact Lie group G on a sphere with exactly one fixed point, we conclude that G is an Oliver group. In fact, the Slice Theorem allows us to remove from the sphere a small invariant open disk around the single fixed point. As a result, we obtain a smooth fixed point free action of G on the disk of dimension of the sphere.

Laitinen and Morimoto [65] proved that any finite Oliver group G has a smooth action on a sphere with exactly one fixed point, and thus they obtained the following theorem.

Theorem 3.2. ([65]) *A finite group G has a smooth action on a sphere with exactly one fixed point if and only if G is an Oliver group.*

Theorems 2.2 and 3.2 yield immediately the following corollary.

Corollary 3.3. ([65], [91]) *If G is a finite group, the following statements are equivalent.*

(1) *G has a smooth action on a sphere with exactly one fixed point.*
(2) *G has a smooth action on a disk without fixed points.*
(3) *G is an Oliver group.*

Except for the two cases considered by Petrie [107] where $G = S^3$ or $SO(3)$, the following question remains open. *Is it true that any non-finite compact Oliver group G has a smooth action on a sphere with exactly one fixed point?* If the answer to this question is affirmative, Corollary 3.3 holds for any compact Lie group G.

4. Fixed point sets up to homotopy type

In this section, we consider actions of compact Lie groups G, where the groups G are not from Smith theory. Recall that a compact Lie group G is not from Smith theory if G_0 is nonabelian or G/G_0 is not of prime power order (cf. Section 2).

For any compact Lie group G which is not from Smith theory, the proof of Theorem 2.1 provides a finite dimensional contractible G-CW complex E such that $E^G = \varnothing$. Moreover, E is infinite and countable (i.e., E has infinitely and countably many G-cells) and E has finitely many orbit types. Now, consider a finite CW complex F. Then the join $X = E * F$ admits the structure of an infinite, countable, finite dimensional, contractible G-CW complex with finitely many orbit types, such that X^G is homeomorphic to F. Thus, Theorem 2.1 obtained for $F = \varnothing$ yields the following theorem (cf. [15, pp. 61–62]).

Theorem 4.1. *Let G be a compact Lie group which is not from Smith theory. Then there exists an infinite, countable, finite dimensional, contractible G-CW complex X with finitely many orbit types, such that X^G is homeomorphic to any given finite CW complex F.*

For a compact Lie group G which is not from Smith theory, and a finite CW complex F, take a G-CW complex X with $X^G = F$, and with the properties described in Theorem 4.1. Then the embedding theorem of Mostow [90] and Palais [95] (cf. [15, Thm. 10.1, p. 111]) allows us to thicken up X into an open contractible smooth G-manifold E in such a way that the manifold E can be chosen to be a Euclidean space. The thickening procedure does not ensure that E^G is homeomorphic to F, but ensures that E^G has the homotopy type of F. As a result, Theorem 4.1 yields the next theorem (cf. [15, Thms. 8.3 and 8.4, pp. 61–62]). Recall that the case where $F = \varnothing$ is already described in Theorem 2.1.

Theorem 4.2. *Let G be a compact Lie group which is not from Smith theory. Then there exists a smooth action of G on some Euclidean space E such that E^G has the homotopy type of any given finite CW complex F.*

Theorem 4.1 asserts that if G is a compact Lie group which is not from Smith theory, then there is no obstruction for a finite CW complex F to occur as the fixed point set of an infinite, countable, finite dimensional, contractible G-CW complex X. If we wish to look for a finite contractible G-CW complex X, the corresponding assertion is not true in general, and it turns out that the Euler characteristic $\chi(F)$ of F is the only occurring obstruction. More precisely, the following results are obtained by Oliver [91]–[94].

For any finite group G not of prime power order, Oliver [91] has proved that the set

$$\{\chi(X^G) - 1 : \ X \text{ is a finite contractible } G\text{-CW complex}\}$$

is a subgroup of the group of integers $\mathbb{Z}$, and thus it has the form $n_G \cdot \mathbb{Z}$ for some unique integer $n_G \geq 0$. Oliver [91], [92] has also proved that the integer n_G exists for any compact Lie group G such that either G_0 is nonabelian (then $n_G = 1$) or G_0 is abelian and G/G_0 is not of prime power order (then $n_G = n_{G/G_0}$). Moreover, Oliver [91]–[93] has computed n_G for any finite group G not of prime power order. Therefore, n_G is defined (and computed) for any compact Lie group G which is not from Smith theory. Henceforth, we refer to n_G as to the *Oliver integer* of G. The main result related with the Oliver integer n_G is contained in the following theorem (cf. Theorem 4.1).

Theorem 4.3. ([91], [92]) *Let G be a compact Lie group which is not from Smith theory. Then the following two statements are true.*

(1) *There exists a finite contractible G-CW complex X such that X^G is homeomorphic to a given finite CW complex F if and only if $\chi(F) \equiv 1 \pmod{n_G}$.*
(2) *There exists a finite contractible G-CW complex X such that X^G is homeomorphic to any given finite CW complex F if and only if $n_G = 1$.*

In Theorem 4.3, (1) implies (2). The following statement also holds: *There exists a finite contractible G-CW complex X such that $X^G = \varnothing$ if and only if $n_G = 1$.*

By using thickening which employs the embedding theorem of Mostow [90] and Palais [95], Oliver [91] converts any finite contractible G-CW complex X obtained in Theorem 4.3 into a compact contractible smooth G-manifolds D such that D^G has the homotopy type of X^G and D can be chosen to be a disk. As a result, he obtains the next theorem (cf. Theorem 4.2). Recall that the case where $F = \varnothing$ is already described in Theorem 2.2.

Theorem 4.4. ([91], [92]) *Let G be a compact Lie group which is not from Smith theory. Then the following two statements are true.*

(1) *There exists a smooth action of G on a disk D such that D^G has the homotopy type of a given finite CW complex F if and only if $\chi(F) \equiv 1 \pmod{n_G}$.*
(2) *There exists a smooth action of G on a disk D such that D^G has the homotopy type of any given finite CW complex F if and only if $n_G = 1$.*

We refer to [94, Thm. 0.3] for a summary of computation of the Oliver integer n_G. Here, we note that if G is a finite group, the condition that G is an Oliver group is equivalent to the condition that G is not of prime power order and $n_G = 1$. As already noted, $n_G = 1$ for any compact Lie group G such that G_0 is nonabelian. Therefore, a compact Lie group G is an Oliver group if and only if G is not from Smith theory and $n_G = 1$.

Following [86], a finite group G is called *pseudocyclic* if G has a normal subgroup P of prime power order, such that the quotient group G/P is cyclic. It turns out that for a finite group G not of prime power order, $n_G = 0$ if and only if G is pseudocyclic, and if the group G is abelian (more generally, nilpotent), then n_G depends on the number of noncyclic Sylow subgroups of G to the effect that $n_G = 0$ when G has no noncyclic Sylow subgroup, $n_G = 1$ when G has three or more noncyclic Sylow subgroups, and $n_G = pq$ when G has exactly two noncyclic Sylow subgroups G_p and G_q for two distinct primes p and q.

5. Connected components of fixed point sets

Bredon [15, p. 58] has remarked that one may well conjecture that for actions of compact groups G on disks, spheres, and Euclidean spaces, each connected component of the fixed point set has the same dimension. The article [97] gives for the first time counterexamples to this conjecture, and [97] also proves that for smooth actions of compact Lie groups G, such counterexamples exist if and only if G/G_0 has an element not of prime power order.

Theorem 5.1. ([97]) *A compact Lie group G has a smooth action on some Euclidean space (resp., disk, sphere) E such that E^G has connected components of different dimensions if and only if G/G_0 has an element not of prime power order.*

For $G = \mathbb{Z}_{pq}$, where p and q are two distinct primes, [97] constructs a smooth action of G on a disk such that the fixed point set has connected components of different dimensions. Then, in the case G/G_0 has an element not of prime power order, [97] uses the results in Propositions 1.1 and 1.2 to obtain a similar action of G on a disk, proving the sufficiency in Theorem 5.1. For actions on spheres and Euclidean spaces, the sufficiency follows by taking the action of G on the double and the interior of the disk, respectively.

Let G be a compact Lie group and let M be a smooth G-manifold with fixed point set F. For any point $x_0 \in F$, the tangent space $T_{x_0}(M)$ is regarded as a real G-module by taking the derivatives (at the point x_0) of the transformations $g : M \to M$, $x \mapsto gx$ for all $g \in G$.

The following result is presented in [97, Prop. 7.1 and Prop. 7.2].

Proposition 5.2. ([97]) *Let G be a compact Lie group such that each element of G/G_0 is of prime power order. Let M be a smooth G-manifold. Then for any two points $x_1, x_2 \in M^G$, the G-modules $T_{x_1}(M)$ and $T_{x_2}(M)$ are isomorphic under either of the two conditions:*

(1) *M is a $\mathbb{Z}$-acyclic space, or*
(2) *M is a $\mathbb{Z}$-homology sphere and $M^G \neq \{x_1, x_2\}$.*

In Proposition 5.2, the G-modules $T_{x_1}(M)$ and $T_{x_2}(M)$ are isomorphic, and thus the trivial summands $T_{x_1}(M)^G$ and $T_{x_2}(M)^G$ have the same dimension, and by the Slice Theorem

$$\dim F(x_1) = \dim T_{x_1}(M)^G = \dim T_{x_2}(M)^G = \dim F(x_2)$$

where $F(x_1)$ and $F(x_2)$ are the connected component of F containing x_1 and x_2, respectively. Therefore, in Theorem 5.1, the necessity of the condition that each element of G/G_0 is of prime power order follows from Proposition 5.2.

If $\dim F(x_1) \neq \dim F(x_2)$ for two points $x_1, x_2 \in F$, the G-modules $T_{x_1}(M)$ and $T_{x_2}(M)$ are not isomorphic because $\dim T_{x_1}(M)^G \neq \dim T_{x_2}(M)^G$ by the Slice Theorem.

If $\dim F(x_1) = \dim F(x_2)$, we may still ask whether or not $T_{x_1}(M)$ and $T_{x_2}(M)$ are isomorphic. For $F(x_1) = F(x_2)$, the answer is affirmative because any G-vector bundle over a path in F is isomorphic to the product G-vector bundle. For $F(x_1) \neq F(x_2)$, the answer depends on the group G, the manifold M, and the action of G on M. In the special case where M is a sphere and $F = \{x_1, x_2\}$, the question whether the G-modules $T_{x_1}(M)$ and $T_{x_2}(M)$ are isomorphic goes back to P.A. Smith [118]. During the pass fourty years, the Smith isomorphism question has created a lot of research. In effect, the question has been aswered for many finite groups G. However, the question is still not answered completely (see [23], [71], and [103] for surveys of the related results, and [104] for answers obtained for many finite Oliver groups, including all finite nonabelian simple groups G).

6. Stably complex fixed point sets

A smooth manifold M is called *stably complex* if the tangent bundle τ_M of M is stably isomorphic to a vector bundle which admits a complex structure. This amounts to saying that there exists a smooth embedding of M into some Euclidean space E, such that the normal bundle of the embedding admits a complex structure. In particular, M is orientable and the connected components of M all have the same parity (cf. [25], [43], [57]).

If a smooth manifold is stably parallelizable, it is stably complex. As a $\mathbb{Z}$-acyclic smooth manifold is also stably parallelizable, any $\mathbb{Z}$-acyclic smooth manifold M is stably complex. More generally, any $\mathbb{Z}_2$-acyclic smooth manifold M is stably complex (cf. [43, (3.2)]).

Proposition 6.1. *Let G be a compact Lie group and let M be a smooth G-manifold. Then the fixed point set M^G is a stably complex manifold under either of the two conditions:*

(1) *G/G_0 is of odd order and M^{G_0} is stably complex, or*
(2) *G/G_0 has a normal Sylow 2-subgroup and M^{G_0} is $\mathbb{Z}_2$-acyclic.*

Let G be a compact Lie group and let M be a smooth G-manifold. As the group G/G_0 is finite and acts smoothly on M^{G_0} with $(M^{G_0})^{G/G_0} = M^G$, the proof of Proposition 6.1 reduces to the case G is finite (i.e., $G = G/G_0$). If G is finite and (2) holds, then M^{G_2} is $\mathbb{Z}_2$-acyclic by Smith theory, where G_2 is the normal Sylow 2-subgroup of G. Hence, M^{G_2} is stably complex and the group G/G_2 of odd order acts smoothly on M^{G_2} with $(M^{G_2})^{G/G_2} = M^G$. Thus, the proof of Proposition 6.1 reduces to the case G is finite and (1) holds. As a finite group of odd order is solvable, (1) allows us to assume that $G = \mathbb{Z}_p$ for an odd prime p, and to conclude that the normal bundle of M^G in M admits a complex structure, which in turn shows that M^G is stably complex, as so is M (cf. [25], [43, (3.1) and (3.2)], [57]).

Proposition 6.1 and Smith theory yield the following corollary (cf. [100, Cor. 1.1]).

Corollary 6.2. *Let G be a compact Lie group and let M be a $\mathbb{Z}$-acyclic smooth G-manifold. Then the fixed point set M^G is a stably complex manifold under the condition that G_0 is abelian and G/G_0 is of odd order or has a normal Sylow 2-subgroup.*

If G is a finite cyclic group, then $G_2 \trianglelefteq G$ and Corollary 6.2 asserts that for any smooth action of G on a Euclidean space or disk, the fixed point set is a stably complex manifold. Edmonds and Lee [43] constructed for the first time smooth actions of finite cyclic groups on Euclidean spaces such that the fixed point set is diffeomorphic to any given closed stably complex smooth manifold whose connected components all have the same dimension.

Theorem 6.3. ([43]) *Let $G = \mathbb{Z}_{pq}$ for two relatively prime integers $p, q \geq 2$, and let F be a closed stably complex smooth manifold such that the connected components of F all have the same dimension. If p and q are sufficiently large with respect to* $\dim F$, *then there exists a smooth action of G on some Euclidean space E such that E^G is diffeomorphic to F.*

Similar smooth actions of finite cyclic groups on disks are described in [96].

Theorem 6.4. ([96]) *Let $G = \mathbb{Z}_{pq}$ for two relatively prime integers $p, q \geq 2$, and let F be a compact stably complex smooth manifold such that the connected components of F all have the same dimension and $\chi(F) = 1$. If p and q are sufficiently large with respect to* $\dim F$, *then there exists a smooth action of G on a disk D such that D^G is diffeomorphic to F.*

7. Fixed point sets of actions of groups from Smith theory

The celebrated results of P.A. Smith ([116], [117]) grew into *Smith theory* (see, e.g., [4], [15], [34], [61]) which assert that under appropriate conditions on a G-space X, homological properties of X are possessed also by the fixed point set X^G in the case where G is a p-group, p-torus, or a torus; i.e., G is from Smith theory (cf. Section 2). By definition, for a prime p, a *p-torus* is an extension of a p-group by a torus $S^1 \times \cdots \times S^1$, a product of circles.

Theorem 7.1. *Let G be a group from Smith theory. Let F be a countable finite dimensional (resp., finite) CW complex. Then F is homeomorphic to the fixed point set of a countable finite dimensional (resp., finite) contractible G-CW complex X with finitely many orbit types if and only if:*

(1) *G is a p-group or p-torus: F is $\mathbb{Z}_p$-acyclic.*
(2) *G is a torus: F is $\mathbb{Z}$-acyclic.*

In Theorem 7.1, in both cases (1) and (2), the necessity of the homological condition on the CW complex F follows from Smith theory.

In case (1), for $G = \mathbb{Z}_p$, the sufficiency is proven by Jones [57] when F is finite, and when F is infinite, the sufficiency can be proven using the arguments of Oliver [91] (cf. Assadi [5]). For any compact Lie group G, the quotient group G/G_0 has a subgroup isomorphic to $\mathbb{Z}_p$ for some prime p. Therefore, in general, the sufficiency follows from the special case $G = \mathbb{Z}_p$ by the quotient and induction arguments (similar to those in Propositions 1.1 and 1.2).

In case (2), for $G = S^1$, the join $X = F * G$ is simply connected, and X equipped with the join action of G has just two orbit types and admits the structure of a countable finite dimensional (resp., finite) G-CW complex such that $X^G = F$. If F is $\mathbb{Z}$-acyclic, so is X, and since X is simply connected, X is contractible. This proves the sufficiency for $G = S^1$. For any torus G, the sufficiency follows now by the quotient or induction arguments.

Theorem 7.2. *Let G be a group from Smith theory. Let F be a smooth manifold such that $\partial F = \varnothing$ (resp., F is compact). Then F is diffeomorphic to the fixed point set of a smooth action of G on some Euclidean space (resp., disk) if and only if:*

(1) *G is a p-group or p-torus: F is $\mathbb{Z}_p$-acyclic and stably complex.*
(2) *G is a torus: F is $\mathbb{Z}$-acyclic.*

In Theorem 7.2, in both cases (1) and (2), the necessity of the homological condition on the manifold F follows from Smith theory (cf. Theorem 7.1). In case (1), the necessity of the condition that F is stably complex follows from Corollary 6.2. If the manifold F is $\mathbb{Z}$-acyclic or $\mathbb{Z}_2$-acyclic, then F is stably complex (cf. Section 6).

Let p be a prime. The famous result of Jones [57] asserts that any compact $\mathbb{Z}_p$-acyclic stably complex smooth manifold F occurs as the fixed point set of a smooth action of $\mathbb{Z}_p$ on a disk. By Propositions 1.1 and 1.2, the same holds for any compact Lie group G such that G/G_0 has an element of order p, proving the sufficiency in Theorem 7.2 in case (1) for actions on disks. For actions on Euclidean spaces, the sufficiency follows from Theorem 7.1, the equivariant thickening in Section 10, and the equivariant bundle extension in Section 12. In case (2), both for actions on Euclidean spaces and disks, the sufficiency is proven in [100], and the corresponding arguments are briefly recalled in Sections 10 and 12.

Let $G = \mathbb{Z}_p$ for some prime p. Let X be a finite connected G-CW complex, and assume that X is a $\mathbb{Z}_p$-homology sphere; i.e., X has the $\mathbb{Z}_p$-homology of a sphere. By Smith theory, the fixed point set $F = X^G$ also is a $\mathbb{Z}_p$-homology sphere. A converse to this Smith theory conclusion is obtained by Willson [125] and roughly reads as follows. If F is a finite CW complex such that F is a $\mathbb{Z}_p$-homology sphere, then F is homeomorphic to the fixed point set of a finite simply connected G-CW complex X such that X is a $\mathbb{Z}_p$-homology sphere. Willson's result follows by extending Jones' methods obtained in the category of finite G-CW complexes (see [125, Theorem 1 and Lemmas A, B, and C] for details).

Let $G = \mathbb{Z}_p$ for some prime p. Let F be a closed smooth manifold of dimension $d \geq 0$. Assume that there exists a smooth action of G on a sphere S such that $S^G = F$. Then Smith theory implies that F is a $\mathbb{Z}_p$-homology sphere. If p is odd, the normal bundle ν of F in S admits a complex structure, and thus F is a stably complex manifold (cf. Proposition 6.1). Ewing [45] proved that the rational Chern classes of ν vanish for $d \neq 2$. Schultz [111] found one more restriction on F to the effect that some invariant (depending on p and d) vanishes provided the codimension of F is S is sufficiently large. In contrast, Schultz's invariant may not vanish without the codimension condition on F in S. Schultz's invariant generalizes the invariant of Kervaire and Milnor [62] for homotopy spheres, and its vanishing means that F is null-cobordant in some sense. It turns out that these restrictions on F are both necessary and sufficient for a closed smooth manifold F to occur as the fixed point set of a smooth action of G on a sphere, where $G = \mathbb{Z}_p$ for any odd prime p.

We refer the reader to [67], [109]–[113], and [123, pp. 285–286] for more precise information, as well as for some discussion about the cases $G = S^1$ or $\mathbb{Z}_{p^a}$ for any integer $a \geq 1$.

In general, if G is a finite p-group, a p-torus, or a torus, the following question is open. *Which smooth manifolds F occur as the fixed point sets of smooth actions of G on spheres?* Again, Smith theory implies that F must be a $\mathbb{Z}_p$-homology sphere when G is a finite p-group or a p-torus, and F must be a $\mathbb{Z}$-homology sphere when G is a torus.

8. Fixed point sets of actions on Euclidean spaces and disks

In this section, for a finite group G not of prime power order, we answer the question which smooth manifolds F occur as the fixed point sets of smooth actions of G on Euclidean spaces or disk. In the case G is a compact Lie group from Smith theory, the answer is given in Theorem 7.2. In general, for a compact non-finite Lie group G not from Smith theory, the answer to the question above is unknown (see [94] for some related information).

For any finite group G not of prime power order, the question above has been answered by Oliver [94]. We emphasize that whereas Theorems 4.2 and 4.4 describe the fixed point sets up to homotopy type, the results below (Theorems 8.2 and 8.3) determine them up to diffeomorphism. Moreover, if F does occur as the fixed point set, Oliver [94] has answered the question of which real G-vector bundles ν over F stably occur as the equivariant normal bundles $\nu_{F \subset E}$ of F in E for smooth actions on Euclidean spaces or disks E with $E^G = F$. More precisely, for any real G-vector bundle ν over F with $\dim \nu^G = 0$, he has determined necessary and sufficient conditions for the existence of a smooth action of G on a Euclidean space (resp., disk) E with $E^G = F$ and such that $\nu_{F \subset E} \cong \nu \oplus \varepsilon_F^V$ for some real G-module V, where ε_F^V denotes the product G-vector bundle over F with fiber V. Some partial related results were obtained earlier in [5], [6], [7], [36], [43], [44], and [96]–[102].

Let G a finite group G not of prime power order. Let F be a countable finite dimensional CW complex with the trivial action of G, and let η be a real G-vector bundle over F. Before we restate Oliver's results, we wish to introduce the notion of Oliver obstruction of η.

Let $[\mathrm{Res}^G_{\{e\}}(\eta)]$ be the element of the reduced KO-theory $\widetilde{KO}(F)$ of F determined by the vector bundle $\mathrm{Res}^G_{\{e\}}(\eta)$ obtained from η by restricting the action of G to the action of the trivial subgroup $\{e\}$ of G. For each prime $p \,\big|\, |G|$ and each p-subgroup P of G with $P \neq \{e\}$, consider the P-vector bundle $\mathrm{Res}^G_P(\eta)$ obtained from η by restricting the action of G to P. Next, take $\widetilde{KO}_P(F)_{(p)}$, the reduced P-equivariant KO-theory of F localized at p, and take its subgroup p-div consisting of all infinitely p-divisible elements. Finally, let $[\mathrm{Res}^G_P(\eta)]$ be the element of the quotient group $\widetilde{KO}_P(F)_{(p)}/p\text{-div}$ determined by $\mathrm{Res}^G_P(\eta)$. Now, define the *Oliver obstruction* $\mathcal{O}\ell(\eta)$ of η by setting

$$\mathcal{O}\ell(\eta) = [\mathrm{Res}^G_{\{e\}}(\eta)] + \sum_{P\neq\{e\}} [\mathrm{Res}^G_P(\eta)] \ \in\ \widetilde{KO}(F) \oplus \bigoplus_{P\neq\{e\}} \widetilde{KO}_P(F)_{(p)}/p\text{-div}$$

where $P \neq \{e\}$ runs over all p-subgroups of G for all primes $p \,\big|\, |G|$.

Theorem 8.1. ([94]) *Let G be a finite group not of prime power order. Let F be a countable finite dimensional (resp., finite) CW complex and let η be a real G-vector bundle over F. Then the following two statements are equivalent.*

(1) *There exists a countable finite dimensional (resp., finite) contractible G-CW complex X such that $X^G = F$, and there exists a real G-vector bundle ξ over X such that as G-vector bundles, $\xi|_F \cong \eta \oplus \varepsilon^V_F$ for some real G-module V.*
(2) $\mathcal{O}\ell(\eta) = 0$ *(resp., $\chi(F) \equiv 1 \pmod{n_G}$ and $\mathcal{O}\ell(\eta) = 0$).*

Given a smooth manifold F and a real G-vector bundle ν over F, we may consider the Whitney sum $\tau_F \oplus \nu$, where τ_F is the tangent bundle of F with the trivial action of G, and take the corresponding Oliver obstruction $\mathcal{O}\ell(\tau_F \oplus \nu)$. By using the equivariant thickening described in Section 10, Theorem 8.1 yields the following theorem.

Theorem 8.2. ([94]) *Let G be a finite group not of prime power order. Let F be a smooth manifold, and assume that $\partial F = \varnothing$ (resp., F is compact). Let ν be a real G-vector bundle over F such that $\dim \nu^G = 0$. Then the following two statements are equivalent.*

(1) *There exists a smooth action of G on some Euclidean space (resp. disk) E such that $E^G = F$ and as G-vector bundles, $\nu_{F\subset E} \cong \nu \oplus \varepsilon^V_F$ for some real G-module V.*
(2) $\mathcal{O}\ell(\tau_F \oplus \nu) = 0$ *(resp., $\chi(F) \equiv 1 \pmod{n_G}$ and $\mathcal{O}\ell(\tau_F \oplus \nu) = 0$).*

Consider the reduced K-theory groups $\widetilde{KO}(F)$, $\widetilde{K}(F)$, and $\widetilde{KSp}(F)$ defined by using the real, complex, and quaternion vector bundles over F, respectively, and the maps

$$\widetilde{KO}(F) \xrightarrow{c_{\mathbb{R}}} \widetilde{K}(F) \xrightarrow{q_{\mathbb{C}}} \widetilde{KSp}(F) \quad\text{and}\quad \widetilde{KSp}(F) \xrightarrow{c_{\mathbb{H}}} \widetilde{K}(F) \xrightarrow{r_{\mathbb{C}}} \widetilde{KO}(F)$$

where $c_{\mathbb{R}}$ and $q_{\mathbb{C}}$ are the induction (complexification and quaternization) maps, and where $c_{\mathbb{H}}$ and $r_{\mathbb{C}}$ are the forgetful (complexification and realification) maps.

In [94, Theorem 0.2 (A)–(F), and Lemma 3.1], the class of finite groups G not of prime power order is divided into six mutually disjoint classes defined by using some G-modules, as well as by using only group theoretic terms. We recall here the latter description. First, we say that a finite group G has a *pq-element* if G has an element of order pq for two distinct primes p and q. Moreover, we say that G has a *pq-dihedral subquotient* if G has subgroups H and N such that $N \trianglelefteq H$ and H/N is isomorphic to the dihedral group of order $2pq$ for two distinct primes p and q. Also, we denote by G_2 any Sylow 2-subgroup of G.

Now, we define six mutually disjoint classes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$ of finite groups G by assuming that G is in the given class if and only if:

$\mathcal{A}$: G has a pq-dihedral subquotient.
$\mathcal{B}$: G has no pq-dihedral subquotient, G has a pq-element conjugate to its inverse.
$\mathcal{C}$: G has no pq-element conjugate to its inverse, G has a pq-element, $G_2 \ntrianglelefteq G$.
$\mathcal{D}$: G has no pq-element conjugate to its inverse, G has a pq-element, $G_2 \trianglelefteq G$.
$\mathcal{E}$: G has no pq-element, $G_2 \ntrianglelefteq G$, G is not of prime power order.
$\mathcal{F}$: G has no pq-element, $G_2 \trianglelefteq G$, G is not of prime power order.

If $G \in \mathcal{A} \cup \mathcal{B}$, then G has a pq-element conjugate to its inverse, and thus $G_2 \ntrianglelefteq G$. In effect, for a finite group G not of prime power order, $G_2 \trianglelefteq G$ if and only if $G \in \mathcal{D} \cup \mathcal{F}$.

Following [94], for an abelian group A, denote by $\mathrm{qdiv}(A)$ the subgroup of quasidivisible elements of A; i.e., the intersection of all kernels of homomorphisms from A into free abelian groups. Also, recall that if A is finitely generated, then $\mathrm{qdiv}(A)$ is just the torsion subgroup $\mathrm{Tor}(A)$ of A. In particular, $\mathrm{qdiv}(\widetilde{KO}(F)) = \mathrm{Tor}(\widetilde{KO}(F))$ and $\mathrm{qdiv}(\widetilde{K}(F)) = \mathrm{Tor}(\widetilde{K}(F))$ for any compact smooth manifold or (more generally) finite CW complex F.

Theorem 8.3. ([94]) *Let G be a finite group not of prime power order. Then a smooth manifold F is diffeomorphic to the fixed point set of a smooth action of G on some Euclidean space (resp., disk) if and only if $\partial F = \varnothing$ (resp., F is compact and $\chi(F) \equiv 1 \pmod{n_G}$) and for the class $[\tau_F]$ in $\widetilde{KO}(F)$ represented by the tangent bundle τ_F the following holds:*

$G \in \mathcal{A}$: *there is no restriction on* $[\tau_F]$.
$G \in \mathcal{B}$: $c_{\mathbb{R}}([\tau_F]) \in c_{\mathbb{H}}(\widetilde{KSp}(F)) + \mathrm{qdiv}(\widetilde{K}(F))$.
$G \in \mathcal{C}$: $[\tau_F] \in r_{\mathbb{C}}(\widetilde{K}(F)) + \mathrm{qdiv}(\widetilde{KO}(F))$.
$G \in \mathcal{D}$: $[\tau_F] \in r_{\mathbb{C}}(\widetilde{K}(F))$*; that is, F is stably complex.*
$G \in \mathcal{E}$: $[\tau_F] \in \mathrm{qdiv}(\widetilde{KO}(F))$.
$G \in \mathcal{F}$: $[\tau_F] \in r_{\mathbb{C}}(\mathrm{qdiv}(\widetilde{K}(F)))$.

While Theorem 8.2 leaves it unclear which smooth manifolds F occur as the fixed point sets of smooth actions of G on Euclidean spaces (resp., disks), Theorem 8.3 provides a very transparent answer to the question for any finite group G not of prime power order.

9. Fixed point sets of $\mathcal{P}$-typical actions on spheres

In this section, we deal with the question which closed smooth manifolds F occur as the fixed point sets of smooth actions of G on spheres for a given finite Oliver group G.

Let G be a finite group not of prime power order. Denote by $\mathcal{P}(G)$ the family of subgroups of G consisiting of the trivial subgroup and all subgroups of prime power order.

An action of G on a space X is called *$\mathcal{P}$-typical* if $X^P \smallsetminus X^G \neq \varnothing$ for each $P \in \mathcal{P}(G)$, which amounts to saying that $X^{G_p} \smallsetminus X^G \neq \varnothing$ for each Sylow p-subgroup G_p of G, $p \,\big|\, |G|$.

Any smooth action of G on a disk D with nonempty fixed point set F can be converted into a smooth $\mathcal{P}$-typical action of G on some disk with the same fixed point set F. In fact, take a real G-module V such that $\dim V^G = 0$ and $\dim V^P > 0$ for each $P \in \mathcal{P}(G)$. Then the diagonal action of G on $D \times D(V)$ is smooth, $\mathcal{P}$-typical, and $(D \times D(V))^G = F$.

Let G be a finite Oliver group. By Smith theory, any smooth fixed point free action of G on a disk is $\mathcal{P}$-typical, and so is any smooth one fixed point action of G on a sphere.

Proposition 9.1. ([87]) *Let G be a finite group not of prime power order, and let $F = S^G$ for a smooth $\mathcal{P}$-typical action of G on a sphere S. Then the Oliver obstruction of $\tau_F \oplus \nu$ vanishes, $\mathcal{O}\ell(\tau_F \oplus \nu) = 0$, where ν is the equivariant normal bundle of F in S.*

Now, for any finite Oliver group G and any closed smooth manifold F, we wish to consider the following two conclusions about the group G and the manifold F.

(S) F occurs as the fixed point set of a smooth $\mathcal{P}$-typical action of G on a sphere.
(D) F occurs as the fixed point set of a smooth $\mathcal{P}$-typical action of G on a disk.

Recall that (D) holds if and only if F occurs as the fixed point set of a smooth action of G on a disk. By Proposition 9.1 and Theorem 8.2, (S) implies (D). Under some conditions imposed on G and F, the article [87] confirms that (D) implies (S), and thus (S) and (D) are equivalent. In general, the question whether (D) implies (S) remains open.

Theorem 9.2. ([87]) *Let G be a finite nontrivial perfect group. Let F be a closed smooth manifold, and assume that each connected component of F is simply connected or stably parallelizable. Then the two conclusions (S) and (D) are equivalent.*

Any finite (nontrivial) perfect group G belongs to one of the classes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, or $\mathcal{E}$ defined in Section 8, and Theorems 9.2 and 8.3 answer the question which closed smooth manifolds (whose connected components are simply connected or stably parallelizable) occur as the fixed point sets of smooth $\mathcal{P}$-typical actions of G on spheres.

We say that a finite group G has a *pqr-cyclic quotient* if G has a cyclic quotient of order pqr for three distinct primes p, q, and r. Recall that a finite group G is nilpotent if and only if G is the product of its Sylow subgroups; i.e., all Sylow subgroups of G are normal. In particular, any finite nilpotent Oliver group G has a pqr-cyclic quotient and $G_2 \trianglelefteq G$.

Theorem 9.3. ([87]) *Let G be a finite Oliver group with a pqr-cyclic quotient. If G is of even order, assume also that $G_2 \trianglelefteq G$. Let F be a closed smooth manifold, and assume that each connected component of F is simply connected or stably parallelizable. Then (S) holds if and only if F is stably complex, and the two conclusions (S) and (D) are equivalent.*

In Theorems 9.2 and 9.3, there is a restriction on the fixed point set F to the effect that each connected component of F is simply connected or stably papallelizable. The restriction is imposed due to the methods applied for constructing smooth actions of G on spheres with prescribed fixed point set F (cf. Sections 14 and 15).

The question which configurations of Pontrjagin numbers occur for the fixed point sets of smooth actions of G on spheres was considered for the first time by Schultz [115] in the case $G = \mathbb{Z}_{pq}$ for two relatively prime odd integers p and q. By [102], all configurations of Chern and Pontrjagin numbers do occur when G is a finite perfect group with appropriate cyclic subgroups. According to [87], the same is true when G is a finite Oliver group with a pqr-cyclic quotient or G is a finite perfect group with a pq-element.

Theorem 9.4. ([87]) *Let G be a finite Oliver group with a pqr-cyclic quotient or let G be a finite perfect group with a pq-element. Let F be a closed oriented smooth manifold of dimension $2k$ (resp., $4k$) for an integer $k \geq 0$. Then there exists a smooth action of G on a sphere such that the fixed point set is a closed oriented smooth manifold of dimension $2k$ (resp., $4k$) with the same Chern (resp., Pontrjagin) numbers as does F.*

10. Equivariant thickening

For a compact Lie group G, constructions of smooth actions of G on Euclidean spaces or disks with prescribed fixed point sets F make use of equivariant thickening procedures. According to [91], the Palais-Mostow embedding theorem yields an equivariant thickening which allows us to describe F up to homotopy type (cf. Theorems 4.2 and 4.4). In order to determine F up to diffeomorphism, we use an equivariant thickening with some equivariant vector bundle data (cf. [5], [43], [99], [100]). Roughly saying, given a smooth manifold F, we look for a G-CW complex X and a smooth G-manifold M such that $M^G = F$ and such that $X \supset M$ as a G-invariant subcomplex. Then we thicken up X into a smooth G-manifold E of the G-homotopy type of X, where $E \supset M$ as a G-invariant submanifold with equivariant normal bundle $\nu_{M \subset E}$ stably isomorphic to a given real G-vector bundle ν over M. This is done by making use of a real G-vector bundle ξ over X such that $\xi|_M \cong \tau_M \oplus \nu \oplus \varepsilon_M^W$ for an appropriate real G-module W (see Theorem 10.4 and Remarks 10.5 and 10.6).

For a G-space X, denote by $\mathcal{F}_{\mathrm{iso}}(G;X)$ the family of the isotropy subgroups occurring at points $x \in X$. Recall that ε_X^V denotes the product G-vector bundle over X whose fiber V is a real G-module. We omit X and write ε^V instead of ε_X^V when X is clear from the context. When we write $b\mathbb{R}$ for some integer $b \geq 0$, we mean that $b\mathbb{R}$ is the direct sum G-module $\mathbb{R} \oplus \cdots \oplus \mathbb{R}$, b times, where G acts trivially on $\mathbb{R}$, and where $b\mathbb{R} = \{0\}$ for $b = 0$.

We denote by $S(V)$ the invariant unit sphere in V, and for a real G-vector bundle ν over a smooth G-manifold M, we denote by $S(\nu)$ the invariant unit sphere bundle of ν.

In the following theorem, we present an equivariant thickening developed in [99] and [100]. For a finite group G, the result is presented in [86, Theorem 3.1] (cf. [5] and [43]).

Theorem 10.1. ([99], [100]) *Let G be a compact Lie group. Let M be a smooth G-manifold with finitely many orbit types, let ν be a real G-vector bundle over M, let X be a countable finite dimensional G-CW complex with finitely many orbit types, and let ξ be a real G-vector bundle over X. Assume that the following three conditions hold.*

(1) *$X \supset M$ as a G-invariant subcomplex.*
(2) *$\mathcal{F}_{\mathrm{iso}}(G; X \smallsetminus M) \subset \mathcal{F}_{\mathrm{iso}}(G; S(V))$ for a real G-module $V \neq \{0\}$.*
(3) *$\xi|_M \cong \tau_M \oplus \nu \oplus \varepsilon^{\ell V} \oplus \varepsilon^{b\mathbb{R}}$ for some integers $\ell \geq 1$ and $b \geq 0$.*

If the integer ℓ is sufficiently large, then there exists a smooth G-manifold E such that the following three conclusions hold.

(4) *$E \supset M$ as a G-invariant submanifold, and $\nu_{M \subset E} \cong \nu \oplus \varepsilon^{\ell V}$.*
(5) *For each $H \in \mathcal{F}_{\mathrm{iso}}(G; E \smallsetminus M)$, there exists $K \in \mathcal{F}_{\mathrm{iso}}(G; S(\nu)) \cup \mathcal{F}_{\mathrm{iso}}(G; S(V))$ such that H is conjugate to a subgroup of K.*
(6) *$E \supset X$ as a G-invariant subcomplex, and there exists a G-deformation retraction $f : E \to X$ such that $\tau_E \oplus \varepsilon^{b\mathbb{R}} \cong f^*(\xi)$. In particular, $\tau_E|_X \oplus \varepsilon^{b\mathbb{R}} \cong \xi$.*

Remark 10.2. In Theorem 10.1, set $d = \dim G$. By replacing V by the $(d+1)$-fold of V, we may assume that $\mathcal{F}_{\mathrm{iso}}(G; S(V))$ is closed under intersection ([100, p. 281]), and if we know that $\mathcal{F}_{\mathrm{iso}}(G; S(\nu)) \subset \mathcal{F}_{\mathrm{iso}}(G; S(V))$, we also know that $\mathcal{F}_{\mathrm{iso}}(G; E \smallsetminus M) \subset \mathcal{F}_{\mathrm{iso}}(G; S(V))$.

Remark 10.3. In Theorem 10.1, assume that $X^G = M^G = F$ and that M is the union of F and all equivariant 0-cells of X, where $F = \varnothing$ or F is a smooth manifold with the trivial action of G. If $\dim(\nu|_F)^G = 0$ and $\dim V^G = 0$, then (5) implies that $E^G = F$.

Theorem 10.1 and Remarks 10.2 and 10.3 yield the following theorem which allows us to construct smooth actions of G on Euclidean spaces and disks with prescribed fixed point sets F, including $F = \varnothing$ (cf. [99] and [100, Section 3]).

Theorem 10.4. ([99], [100]) *Let G be a compact Lie group. Let $F = \varnothing$ or let F be a smooth manifold, let ν be a real G-vector bundle over F, let X be a countable finite dimensional G-CW complex with finitely many orbit types, and let ξ be a real G-vector bundle over X. Assume that the following three conditions hold.*

(1) *$X^G = M^G = F$ where M is the union of F and all equivariant 0-cells of X.*
(2) *$\mathcal{F}_{\text{iso}}(G; X \smallsetminus F) \cup \mathcal{F}_{\text{iso}}(G; S(\nu)) \subset \mathcal{F}_{\text{iso}}(G; S(V))$ for a real G-module $V \neq \{0\}$.*
(3) *$\xi|_F \cong \tau_F \oplus \nu \oplus \varepsilon^{\ell V} \oplus \varepsilon^{b\mathbb{R}}$ for some integers $\ell \geq 1$ and $b \geq 0$.*

Assume also that $\dim \nu^G = 0$ and $\dim V^G = 0$. If the integer ℓ is sufficiently large, then there exists a smooth G-manifold E such that the following three conclusions hold.

(4) *$E \supset M$ as a G-invariant submanifold, $E^G = M^G = F$, and $\nu_{F \subset E} \cong \nu \oplus \varepsilon^{\ell V}$.*
(5) *$\mathcal{F}_{\text{iso}}(G; E \smallsetminus F) \subset \mathcal{F}_{\text{iso}}(G; S(V))$.*
(6) *The conclusion (6) of Theorem 10.1 holds.*

Remark 10.5. In Theorem 10.4, assume that X is infinite and $\partial F = \varnothing$. Then the resulting smooth G-manifold E is open (i.e., E is not compact and $\partial E = \varnothing$). If X is contractible, then so is E because $E \simeq X$ by (6). As we can construct E in such a way that E is simply connected at infinity, we use the results of Stallings [119] to conclude that E can be chosen to be the Euclidean space of dimension $n = \dim(\tau_F \oplus \nu) + \ell \dim V \geq 5$.

Remark 10.6. In Theorem 10.4, assume that X is finite ($\partial F = \varnothing$ or $\partial F \neq \varnothing$). Then the resulting smooth G-manifold, denoted now by D, is compact. If X is contractible, then so is D because $D \simeq X$ by (6). As we can construct D in such a way that the boundary ∂D is simply connected, we use the h-Cobordism Theorem of Smale [75] to conclude that D can be chosen to be the disk of dimension $n = \dim(\tau_F \oplus \nu) + \ell \dim V \geq 6$.

Problem 10.7. Let G be a compact Lie group. Let F be a smooth manifold, and assume that $\partial F = \varnothing$ (resp., F is compact). Now, the problem is to construct a real G-vector bundle ν over F, a real G-module V, and an infinite (resp., finite) contractible G-CW complex X such that the assumptions of Theorem 10.4 all hold.

Once Problem 10.7 is solved for specific G and F, there exists a smooth action of G on some Euclidean space (resp., disk) with fixed point set F. In fact, the required action exists by Theorem 10.4 and Remark 10.5 (resp., Theorem 10.4 and Remark 10.6).

11. Equivariant extension of G-vector bundles: I

Let G be a compact Lie group. If a G-space X is nonequivariantly contractible, then any real (resp., complex) G-vector bundle over X, is nonequivariantly a product bundle $X \times \mathbb{R}^n$ (resp., $X \times \mathbb{C}^n$) over X. So, in order to construct a real (resp., complex) G-vector bundle ξ over X, we can apply the following procedure which goes back to Bredon [15].

For an integer $n \geq 1$, put $C(n) = O(n)$ or $U(n)$ and consider the space $\text{Map}_*(G, C(n))$ of pointed maps $\theta : G \to C(n)$ with the compact-open topology, where the neutral elements of G and $C(n)$ play the role of the base points. Next, consider the action of G on $\text{Map}_*(G, C(n))$ given by $(g\theta)(h) = \theta(hg)\theta(g)^{-1}$ for all $g, h \in G$. Clearly, $\text{Map}_*(G, C(n))^G = \text{Hom}(G, C(n))$. The following proposition goes back to Bredon [15, VI, Proposition 11.1].

Proposition 11.1. ([15]) *Let G be a compact Lie group. For a G-space X and for $n \geq 1$, there exists a natural one-one correspondence between orthogonal (resp., unitary) G-vector bundle structures on $X \times \mathbb{R}^n$ (resp., $X \times \mathbb{C}^n$) over X and G-maps $\theta : X \to \mathrm{Map}_*(G, C(n))$, where $C(n) = O(n)$ (resp., $C(n) = U(n)$). For a given G-map $\theta : x \mapsto \theta_x$, the corresponding G-action on $X \times \mathbb{R}^n$ (resp., $X \times \mathbb{C}^n$) is given by $g(x, v) = (gx, \theta_x(g) \cdot v)$ for all $g \in G$, $x \in X$ and $v \in \mathbb{R}^n$ (resp., $v \in \mathbb{C}^n$).*

We may apply Proposition 11.1 to perform equivariant extensions of G-vector bundles. In fact, for a given compact Lie group G and a smooth manifold F, we construct a real G-vector bundle ν over F such that $\dim \nu^G = 0$ and $\tau_F \oplus \nu$ is nonequivariantly a product bundle. Then, we consider the G-map

$$\rho : F \to \mathrm{Hom}(G, C(n)) \subset \mathrm{Map}_*(G, C(n)),\ x \mapsto \rho_x$$

where $\rho_x : G \to C(n)$ is the representation of G determined on the fiber of $\tau_F \oplus \nu$ over x. For a given finite contractible G-CW complex X with $X^G = F$, we extend ρ to a G-map $X \to \mathrm{Map}_*(G, C(n))$ to get a G-vector bundle ξ over X such that $\xi|_F \cong \tau_F \oplus \nu$.

In [97] and [101], this approach is applied to obtain a solution of Problem 10.7 and in effect to obain smooth actions of G on Euclidean spaces and disks, and moreover on spheres by doubling of disks. In particular, this equivariant extension method is used in the proof of the sufficiency part of the "if and only if" assertion in Theorem 5.1.

12. Equivariant extension of G-vector bundles: II

Edmonds and Lee [43] have used computations of the equivariant complex K-theory of some G-spaces to obtain a solution of Problem 10.7. Now, we briefly discuss their approach, which in particular is applied to prove the results in Theorems 6.3 and 6.4.

For a prime p and an integer $n \geq 1$, let S_p^{2n-1} be the sphere S^{2n-1} with the standard free actions of $\mathbb{Z}_p$. Then, as rings, $K_{\mathbb{Z}_p}(S_p^{2n-1}) \cong K(S_p^{2n-1}/\mathbb{Z}_p) \cong R(\mathbb{Z}_p)/(1-t)^n$ where $R(\mathbb{Z}_p)$ is the complex representation ring of $\mathbb{Z}_p$. Recall that $R(\mathbb{Z}_p) \cong \mathbb{Z}[t]/(1-t^p)$.

Let $G = \mathbb{Z}_p$ with $p = p_1 \dots p_k$ for distinct primes $p_1, \dots, p_k$ and $k \geq 1$. Consider the join $S = S_{p_1}^{2n-1} * \dots * S_{p_k}^{2n-1}$ with the join action of G. For a finite CW complex F, the join $F * S$ is a finite G-CW complex with $(F * S)^G = F$. Under the assumption that F is a compact stably complex smooth manifold, Edmonds and Lee [43] show that the element of $K(F)$ determined by the complex stable tangent bundle τ_F^{st} lies in the image of the composition $K_G(F * S) \to K_G(F) \to K(F)$ of the restriction map $\xi \mapsto \xi|_F$ and the trivial summand map $\eta \mapsto \eta^G$, provided the primes p_i are sufficiently large with respect to $\dim F$. The article [102] removes this restriction on the primes p_i by showing that for any primes $p_1, \dots, p_k$, there exist integers $a_1, \dots, a_k$ with $a_i \geq 0$ such that the element $p_1^{a_1} \dots p_k^{a_k} \tau_F^{\mathrm{st}}$ of $K(F)$ lies in the image of the composition above (see [102] for applications of the result for constructions of smooth group actions on disks and spheres with prescribed fixed point sets).

Proposition 12.1. ([99]) *Let $G = \mathbb{Z}_p$ for a prime p. Let $S = S_p^{2n-1}$ for an integer $n \geq 1$, and let F be a compact $\mathbb{Z}_p$-acyclic stably complex smooth manifold. Then the composition $K_G(F * S) \to K_G(F) \to K(F)$ is an epimorphism. In particular, the element τ_F^{st} of $K(F)$ lies in the image of the composition $K_G(F * S) \to K_G(F) \to K(F)$.*

Let G and F be as in Proposition 12.1. For any integer $n \geq 1$, Proposition 12.1 shows that there exists a G-vector bundle η over $F * S_p^{2n-1}$ such that $\eta|_F \cong \tau_F \oplus \nu \oplus \varepsilon^{b\mathbb{R}}$ for some

G-vector bundle ν over F with $\dim \nu^G = 0$ and some integer $b \geq 0$. According to Jones [57], there exists a finite contractible G-CW complex X such that $X^G = F$. Obstruction theory yields a G-map $f : X \to F * S_p^{2n-1}$ extending the identity map on F (n sufficiently large). Set $\xi = f^*(\eta)$. Then $\xi|_F \cong \tau_F \oplus \nu \oplus \varepsilon^{b\,\mathbb{R}}$. To obtain a solution of Problem 10.7, take X and ν as above, and let V be any real G-module with $\dim V \geq 1$ and $\dim V^G = 0$. Now, by Theorem 10.4 and Remark 10.6, there exists a smooth action of G on a disk D with $D^G = F$. Recall that such action was obtained for the first time by Jones [57]. Proposition 12.1 allows us to simplify Jones' construction. A similar result holds when F is non-compact. In effect, by using Propositions 1.1 and 1.2, we prove the sufficiency in Theorem 7.2 in case (1).

The sufficiency in case (2) follows easily. In fact, we may assume that $G = S^1$ and argue as follows. The joint $X = F * G$ is contractible and F is stably parallelizable because F is $\mathbb{Z}$-acyclic by assumption (cf. Section 7). Set $\xi = \varepsilon_X^W$ for $W = d\,\mathbb{R} \oplus V$, where $d = \dim F$ and V is a real G-module containing free orbits, and $\dim V^G = 0$. To obtain a solution of Problem 10.7, take X and V as above and put $\nu = \varepsilon_F^V$. By solving Problem 10.7, we obtain a smooth action of G on some Euclidean space (resp., disk) E with $E^G = F$.

13. Equivariant extension of G-vector bundles: III

Following Oliver [94], for a finite group G not of prime power order, consider the classifying space B_GO of real G-vector bundles as the infinite mapping cylinder of maps

$$B_GO(0) \to B_GO(r) \to B_GO(2r) \to B_GO(3r) \to \dots$$

where $r = |G|$ and for any integer $n \geq 0$, $B_G(nr)$ is the classifying space of nr-dimensional real G-vector bundles and the map $B_GO(nr) \to B_GO((n+1)r)$ is stabilization by the real regular G-module $\mathbb{R}[G]$ (cf. [34, Theorem I.8.12]).

Oliver [94] defines a G-space B_G^*O and a G-map $L_G : B_GO \to B_G^*O$, and he proves that L_G is a (nonequivariant) homotopy equivalence [94, Definition 1.1 and Lemma 2.1]. Then he shows how to construct G-maps from a finite G-CW complex into B_G^*O and how to lift such G-maps to B_GO [94, Propositions 1.3 and 2.3]. Next, he constructs G-vector bundles by combining a prescribed G-vector bundle and families of P-vector bundles for all $P \in \mathcal{P}(G)$, satisfying some compatibility [94, Theorem 2.4]. By using Oliver's procedure, we can obtain a solution of Problem 10.7. More specifically, we can proceed as follows.

Let G be a finite group not of prime power order. Let F be a countable CW complex, and assume that F is finite dimensional (resp., finite and $\chi(F) \equiv 1 \pmod{n_G}$). According to [94] (resp., [91]), there exists a countable finite dimensional (resp., finite) contractible G-CW complex Y such that $Y^G = F$. Let $Y_F = (Y/F) \vee F$ be the wedge of the quotient space Y/F and F with respect to the base point $F \in Y/F$ and some base point $x_0 \in F$. Then the following theorem follows from [94, Theorem 2.4] (cf. [86, Theorem 2.1]).

Theorem 13.1. ([94], [86]) *With G, F, Y, and Y_F as above, let η be a real G-vector bundle over F such that the Oliver obstruction $\mathcal{O}\ell(\eta) = 0$. Then there exist a countable contractible G-CW complex X and a real G-vector bundle ξ over X such that X is finite dimensional (resp., finite) and the following two conditions hold.*

(1) *$X^G = F$, $X \supset Y_F$ as a G-invariant subcomplex, and $\mathcal{F}_{\mathrm{iso}}(G; X \smallsetminus Y_F) \subset \mathcal{P}(G)$.*
(2) *$\xi|_F \cong \eta \oplus \varepsilon_F^W$ and $\xi|_{Y/F} \cong \varepsilon_{Y/F}^{V\oplus W}$, where V is the fiber of η over the point $x_0 \in F$ and $W = \ell\,\mathbb{R}[G]$ for some integer ℓ.*

14. Equivariant surgery

For a compact Lie group G, the equivariant thickening described in Section 10 produces smooth actions of G on noncompact manifolds without boundary (e.g., Euclidean spaces) or compact manifolds with boundary (e.g., disks). For the study of smooth actions of G on closed manifolds (e.g., spheres), one usually employs a version of equivariant surgery such as that developed by Petrie [105]–[107] (as well as his students and collaborators [38], [108]) to deal with the following problem. For a closed smooth manifold F and a real G-vector bundle ν over F with $\dim \nu^G = 0$, construct a closed smooth G-manifold S of a given homotopy type, and such that $S^G = F$ and $\nu_{F \subset S} \cong \nu$. When trying to solve the problem, the idea is to construct a normal G-map $f : X \to Y$ from a closed smooth G-manifold X into a closed smooth G-manifold Y of the given homotopy type, where $X^G = F$ and $\nu_{F \subset X} \cong \nu$. Then, by killing of surgery obstructions, one tries to convert f into a G-map $h : S \to Y$ such that h is a (nonequivariant) homotopy equivalence and S is a closed smooth G-manifold with $S^G = X^G = F$ and $\nu_{F \subset S} \cong \nu_{F \subset X} \cong \nu$.

Morimoto [78]–[83] (in part, with the coauthors of the articles [10], [11], [64], [65]) has presented powerful extensions and modifications of Petrie's equivariant surgery techniques. In particular, the "deleting–inserting" theorems are proved in [64, Theorem 2.2] for any finite nonsolvable group G, and in [83, Theorems 0.1 and 4.1] for any finite Oliver group G. These theorems allow us to modify a smooth action of G on a sphere so that to get a new smooth action of G on the same sphere with a new fixed point set obtained from the previous one by deleting or inserting a number of the previous fixed point set connected components. Here, we restate only the "deleting parts" of [83, Theorems 0.1 and 4.1].

For a finite group G, we denote by $\mathcal{S}(G)$ the family of all subgroups of G. Moreover, for a prime p, we denote by G^p the smallest normal subgroup of G such that $|G/G^p| = p^a$ for some integer $a \geq 0$. Following [65], we refer to G^p as to the *Dress subgroup of G of type p*. Now, we set $\mathcal{L}^p(G) = \{H \in \mathcal{S}(G) \mid G^p \leq H\}$ and $\mathcal{L}(G) = \bigcup_p \mathcal{L}^p(G)$. A subgroup H of G is called a *large subgroup* of G if $H \in \mathcal{L}(G)$.

A subgroup H of G is called *pseudocyclic* if H has a normal subgroup P with $P \in \mathcal{P}(G)$, such that H/P is cyclic. Let $\mathcal{PC}(G)$ be the family of all pseudocyclic subgroups of G. According to [65], $\mathcal{PC}(G) \cap \mathcal{L}(G) = \varnothing$ for any finite Oliver group G.

A real G-module V is called *G-oriented* if V^H is oriented for each $H \in \mathcal{S}(G)$, and the transformations $g : V^H \to V^H$ preserve orientation for all $g \in N_G(H)$, where $N_G(H)$ is the normalizer of H in G. If V is the realification of a complex G-module, then V is G-oriented (where the fixed point sets V^H have the canonical orientation determined by the complex structure). A real G-module V is called *$\mathcal{P}$-oriented* if V^P is oriented for each $P \in \mathcal{P}(G)$, and the transformations $g : V^P \to V^P$ preserve orientation for all $g \in N_G(P)$.

For a G-space X, we denote by $\mathcal{F}_{\text{iso}}(G; X)$ the family of the isotropy subgroups G_x which occur at points $x \in X$. For a subgroup H of G, we set $X^{=H} = \{x \in X \mid G_x = H\}$.

Moreover, for a subspace A of X^G, we denote by $\mathcal{L}(X, A)$ the union of the connected components of X^H which intersect A, taken for all large subgroups H of G.

A real G-module V is called *$\mathcal{L}$-free* if $\dim V^H = 0$ for each large subgroup H of G. More generally, a real G-vector bundle ν over a smooth manifold F is called *$\mathcal{L}$-free* if each fiber of ν is $\mathcal{L}$-free as a real G-module. Let M be a smooth G-manifold and let A be a union of connected components of M^G. If the equivariant normal bundle $\nu_{A \subset M}$ is $\mathcal{L}$-free, then $\mathcal{L}(M, A) = A$ by the Equivariant Tubular Neighborhood Theorem.

Theorem 14.1. ([83], [87]) *Let G be a finite Oliver group acting smoothly on a homotopy sphere Σ in such a way that the following five conditions hold.*

(1) *For each $P \in \mathcal{P}(G)$ and $H \in \mathcal{S}(G)$ with $P < H$, $\dim \Sigma^P > 2 \dim \Sigma^H$.*
(2) *For each $P \in \mathcal{P}(G)$ and $H \in \mathcal{PC}(G)$, $\dim \Sigma^P \geq 5$ and $\dim \Sigma^{=H} \geq 2$.*
(3) *For each $P \in \mathcal{P}(G)$, Σ^P is simply connected.*
(4) *For a chosen point $x \in \Sigma^G$, the tangent G-module $T_x(\Sigma)$ is $\mathcal{P}$-oriented.*
(5) *For a chosen decomposition $\Sigma^G = F \sqcup F'$ of Σ^G into the disjoint union of two closed smooth manifolds F and F', $\mathcal{L}(\Sigma, F) \cap \mathcal{L}(\Sigma, F') = \varnothing$.*

Then there exists a smooth action of G on the sphere S such that $\dim S^P = \dim \Sigma^P$ and S^P is simply connected for each $P \in \mathcal{P}(G)$. Moreover, $S^G = F$ and $\nu_{F \subset S} \cong \nu_{F \subset \Sigma}$.

Theorem 14.2. ([83], [87]) *Let G be a finite Oliver group acting smoothly on a disk D in such a way that $\partial F = \varnothing$ where $F = D^G$, and the following five conditions hold.*

(1) *For each $P \in \mathcal{P}(G)$ and $H \in \mathcal{S}(G)$ with $P < H$, $\dim D^P > 2 \dim D^H$.*
(2) *For each $P \in \mathcal{P}(G)$ and $H \in \mathcal{PC}(G)$, $\dim D^P \geq 5$ and $\dim D^{=H} \geq 2$.*
(3) *For each $P \in \mathcal{P}(G)$, D^P is simply connected.*
(4) *The tangent G-module $T_x(D)$ is $\mathcal{P}$-oriented for a chosen point $x \in F$.*
(5) *$\mathcal{L}(D, F) \cap \partial D = \varnothing$.*

Then there exists a smooth G-manifold C diffeomorphic to D such that $C^G = \varnothing$ and C^P is simply connected for each $P \in \mathcal{P}(G)$, and there exists a G-diffeomorphism $f : \partial C \to \partial D$ such that the identification space $S = C \cup_f D$ is the standard sphere.

Theorems 14.1 and 14.2 both hold when the gap condition (1) is replaced by a weaker gap condition (cf. [83]). If $\dim F = 0$, the weaker gap condition can be arranged for any finite Oliver group G. However, if $\dim F > 0$, we are forced to assume that G is a *gap group* as defined in [88] (see [88] and [122] for basic information on gap groups).

In order to construct a smooth action of a finite Oliver group G on a sphere with exactly one fixed point, we choose an appropriate real G-module V with $\dim V^G = 0$ and consider the sphere $S = S(V \oplus \mathbb{R})$ with the linear action of G, where G acts trivially on $\mathbb{R}$. The fixed point set S^G consists of two points one of which is deleted by making use of Theorem 14.1. This procedure is used to prove Theorems 3.1 and 3.2.

In general, for a closed smooth manifold F, one may proceed as follows. First, construct a smooth action of G on a disk D with $D^G = F$ and then consider the equivariant double $D \cup_{\partial D} D$ of D to obtain a smooth action of G on the sphere $S = D \cup_{\partial D} D$ with $S^G = F \sqcup F$. But try to arrange the action of G on D in such a way that the resulting action of G on S satisfies the assumptions of Theorem 14.1. Then, using Theorem 14.1, delete one copy of F from S to get a smooth action of G on S with $S^G = F$. Alternatively, apply Theorem 14.2 to delete F from the disk D, i.e., to obtain both C and f as described in Theorem 14.2, and in effect to obtain a smooth action of G on $S = C \cup_f D$ such that $S^G = F$ and $\nu_{F \subset S} \cong \nu_{F \subset D}$. This procedure is applied to prove Theorems 9.2, 9.3, and 9.4.

In order to arrange the condition (5) in Theorems 14.1 and 14.2, we work with real $\mathcal{L}$-free G-vector bundles ν over F, such that $\xi|_F \cong \tau_F \oplus \nu \oplus \varepsilon^{\ell V}$ for a real G-module V which comes from a solution of Problem 10.7. Usually, V is not $\mathcal{L}$-free, and thus $\nu \oplus \varepsilon^{\ell V}$ is not $\mathcal{L}$-free. In the next section, we discuss a procedure which allows us to overcome this difficulty.

15. Equivariant subtraction of G-vector bundles

The article [86] presents an equivariant subtraction procedure which allows us to subtract a product G-vector bundle summand from a given G-vector bundle. This procedure is useful for obtaining G-vector bundles with prescribed isotropy subgroups.

Let G be a finite group. For a subgroup H of G, a real G-module V is called *H-complete* if the irreducible H-submodules of $\mathrm{Res}^G_H(V)$ form a complete list of the irreducible H-modules. For a family $\mathcal{F}$ of subgroups of G, a real G-module V is called *$\mathcal{F}$-complete* if V is H-complete for each $H \in \mathcal{F}$. For example, the real regular G-module $\mathbb{R}[G]$ is $\mathcal{F}$-complete for $\mathcal{F} = \mathcal{S}(G)$.

Now, we restate the Equivariant Vector Bundle Subtraction Theorem proven in [86].

Theorem 15.1. ([86]) *Let G be a finite group and let $\mathcal{F}$ be a family of subgroups of G. Let U be a real G-module and let V be a real $\mathcal{F}$-complete G-module. Let (X, A) be a pair of finite G-CW complexes such that $X \supset A$ as a G-invariant subcomplex, and $\mathcal{F}_{\mathrm{iso}}(G; X \smallsetminus A) \subset \mathcal{F}$. Let ξ and η be G-vector bundles over X and A, respectively, such that $\xi|_A \cong \eta \oplus \varepsilon^{U \oplus \ell V}$ for an integer $\ell \geq 1$. If ℓ is sufficiently large, there exists a G-subbundle θ of ξ such that $\theta \cong \varepsilon^U_X$ and $(\xi - \theta)|_A \cong \eta \oplus \varepsilon^{\ell V}$ for the G-orthogonal complement $\xi - \theta$ of θ in ξ.*

Laitinen and Morimoto [65] have defined the following real $\mathcal{L}$-free G-module:

$$V(G) = (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_{p\,|\,|G|} (\mathbb{R}[G]^{G^p} - \mathbb{R})$$

where $\mathbb{R}[G]^{G^p}$ has the canonical action of G and the subtracted summands $\mathbb{R}$ have the trivial action of G. In [65], for any finite Oliver group G and any even integer $\ell \geq 6$, the G-module $\ell V(G)$ is used (in the way described in Section 14) to construct a smooth action of G on the sphere S with $\dim S = \ell \dim V(G)$, such that $S^G = \{x\}$ and $T_x(S) \cong \ell V(G)$.

According to [86], the G-module $V(G)$ is $\mathcal{F}$-complete for $\mathcal{F} = \mathcal{S}(G) \smallsetminus \mathcal{L}(G)$. By using Theorems 13.1 and 15.1, we may try to solve Problem 10.7 to obtain $\xi|_F \cong \tau_F \oplus \nu \oplus \varepsilon^{\ell V}$ where the G-vector bundle ν over F and the G-module V both are $\mathcal{L}$-free.

By applying the results of [85] and [86], such a solution of Problem 10.7 is obtained in the article [87] in the case where F is simply connected or stably parallelizable, and the resulting G-CW complex X has the property that X^P is simply connected for each $P \in \mathcal{P}(G)$. Then, by making use of Theorems 10.1 and 14.1, the solution of Problem 10.7 produces a smooth action of G on a sphere S with $S^G = F$. In effect, by combining the equivariant extension, subtraction, thickening and surgery arguments, we prove Theorems 9.2, 9.3, and 9.4.

16. Epilogue

In this article, we have discussed the main methods which allow us to construct smooth actions of compact Lie groups G on Euclidean spaces, disks, and spheres with prescribed manifolds F occurring as the fixed point sets. We have covered many cases of G and F where the corresponding constructions can be performed. However, in many other important cases, much remains to be understood about the applied techniques. In order to work on any of the unsolved cases, we need to obtain equivariant extensions of G-vector bundles defined over F. Such explicit extension is required for equivariant thickening and it is done implicitly once equivariant surgery is performed. The bibliography below contains a number of references which cover material related with the topics discussed in this article.

In the article [51, pp. 224 and 231] Wu-Chung Hsiang and Wu-Yi Hsiang have expressed the following opinion. *Due to the existence of natural linear actions on Euclidean spaces, spheres and disks, it is quite fair to say that they are the best testing spaces in the study of differentiable transformation groups ... We share the prevailing conviction that the study of differentiable actions on these best testing spaces is probably still the most important topic in transformation groups.* During the past century, a lot of results related with this topic have been obtained by many authors (see, e.g., the proceedings [89], [73], [113], [56], [60]). The goal of this article was to give an up to date survey of the results about the fixed point sets of smooth compact Lie group actions on Euclidean spaces, disks, and spheres.

Professor Kawakubo organized the International Conference on Transformation Groups at Osaka University in December, 1987. It was a big honour and pleasure for me to attend the conference and to meet Professor Kawakubo. In 1999, with great regret I have learned that Professor Kawakubo has passed away. I would like to dedicate my article to his memory.

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